#CHAPTER#

#Separation#

A top-space may be very sparsely endowed with open sets. Some space has only two, the empty set and the full space. On the other hand every subset of a discrete space is an open set. Most of the familiar spaces of Geometry and analysis fall somewhere between these artificial extremes. The so-called seperation properties enable us to state precisely that a given top-space has a rich enough supply of open sets to serve our purpose in mind.

The supply of open sets in top-space is intimately linked with the supply of continuous functions. Generally more open sets there are, the more continuous functions a space has. Discrete spaces have the greatest possible numbers of continuous functions, because all functions are continuous. However few really important spaces are discrete, this goes a bit too far. The seperation properties make it possible for us to be sure that our spaces have enough continuous functions.

The definition of topology does not indicate such properties which may distinguish points of the space. In this chapter we discuss some important top-spaces which have open or closed sets satisfying certain conditions in relation to their points.

The seperation axioms identify all these spaces. The real line has most, in fact all, of these axioms. These properties are denoted as

$T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$

Separation By Open Sets

(a) Weak Separation #

Let (X, T) be a topological space. Two non-empty subsets A and B are said to be weakly seperated if there exist two open sets G & H such that

$A \subseteq G$, $B \subseteq H$ $\quad A \cap H = \phi$ and $B \cap G = \phi$

(b) Strong Separation #

Two non-empty sets in a top-space are said to be strongly separated if $\exists$ two open set G and H such that

$A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$

Comments #(i) It is clear from the definitions that strongly separated sets always weakly separated.

(ii) We donot demand that sets A & B be open or closed. Weakly seperated sets also called (simply) seperated sets

Theorem # Two non-empty subsets A & B are weakly separated iff any one of the following conditions holds

(i) # $A \cap B = \phi$, $A \cap B^d = \phi$ $\quad B \cap A^d = \phi$

i.e iff A & B are disjoint and neither of them contains limit points of the other.

(ii) # $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$. (Hausdorff-Lennes Condition)

OR equivalently

$A \cap \bar{B} = \phi$ $\quad \bar{A} \cap B = \phi$

Proof # (i) Let A & B be weakly seperated by open sets G & H. Then

$A \subseteq G$ $\quad B \subseteq H$ $\quad A \cap H = \phi$ & $B \cap G = \phi$

Now $A \subseteq G$ & $B \cap G = \phi \Rightarrow A \cap B = \phi$

Also G is an open set containing A (all points of A) and G does not intersect B. Hence no point of A

can be an accumulation point of B i.e

$$A \cap B^d = \phi$$

Similarly $B \cap A^d = \phi$

Thus condition (i) is true.

Conversely let the condition (i) is true i.e.

$A \cap B = \phi$ $\quad A \cap B^d = \phi$ $\quad B \cap A^d = \phi$

Then

$$\bar{A} \cap B = (A \cup A^d) \cap B$$
$$= (A \cap B) \cup (A^d \cap B)$$
$$= \phi \cup \phi = \phi$$

and similarly $A \cap \bar{B} = \phi$

Hence $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$

Thus (i) $\Longrightarrow$ (ii)

Finally let (ii) holds. Then

$$\bar{A} \cap B = \phi \quad \text{and} \quad A \cap \bar{B} = \phi$$

So that

$$B \subseteq (\bar{A})^c = X - \bar{A} \quad \& \quad A \subseteq X - \bar{B}$$

$\because$ $\bar{A}$, $\bar{B}$ are closed sets

$\therefore$ $X - \bar{A}$ & $X - \bar{B}$ are open sets.

By taking $G = X - \bar{A}$ & $H = X - \bar{B}$

we get

$B \subseteq G$, $A \subseteq H$

$B \subseteq G$ $\quad A \subseteq H$ $\quad B \cap H = \phi$ $\quad A \cap G = \phi$

Hence A & B are weakly seperated.

Comments # Two seperated sets are always disjoint.

However two disjointed sets may not be seperated.

i.e $A \cap B = \phi \not\Rightarrow A \cap \bar{B} = \phi$

e.g Let $A =]-\infty \; 0[$ and $B = [0 \quad +\infty[$

Then $A \cap B = \phi$

However $\bar{A} \cap B =]-\infty \; 0] \cap [0 \; \infty[= \{0\} \neq \phi$

$\Rightarrow$ A & B are not seperated.

Corollary # If C & D are non-empty subsets of two weakly seperated subsets A & B respectively, then C and D are also weakly seperated in (X, T). And similar result holds for strong seperation.

Proof # Since A & B are weakly seperated, therefore there exist open sets G, H such that

$A \subseteq G, \quad B \subseteq H \quad A \cap H = \phi \quad B \cap G = \phi$

Now

$C \subseteq A \subseteq G \qquad D \subseteq B \subseteq H$

$\Rightarrow C \cap H \subseteq A \cap H = \phi$

But $\phi \subseteq C \cap H$

$\Rightarrow C \cap H = \phi$

Similarly $D \cap G \subseteq B \cap G = \phi$

$\Rightarrow D \cap G = \phi$

Thus we have

$C \subseteq G \quad D \subseteq H \quad C \cap H = \phi \quad D \cap G = \phi$

$\Rightarrow C, D$ are also weakly seperated.

Next let A & B be strongly seperated set. Then $\exists$ open sets U & V such that

$A \subseteq U \quad B \subseteq V$ and $U \cap V = \phi$

Now

$C \subseteq A \subseteq U \qquad D \subseteq B \subseteq V$

and $U \cap V = \phi$

$\Rightarrow C, D$ are also strongly separated in (X, T)

Theorem # (Wedge-Theorem)

Two disjoint subsets A & B are seperated in a top-space (X, T) iff both A & B are open and closed in $D = A \cup B$

Proof # Let A & B be seperated, so that by definition

$A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi \qquad A \cap B = \phi$

Let $A \cup B = D$ and let $\bar{A}_D$ be closure of A in set D

Now $\bar{A}_D = \bar{A} \cap D = \bar{A} \cap (A \cup B)$

$= (\bar{A} \cap A) \cup (\bar{A} \cap B)$

$= A \cup \phi = A$ $\quad A \subseteq \bar{A}$

$\Rightarrow$ the set A is closed in $D = A \cup B$ & $\bar{A} \cap B = \phi$ given.

Similarly set B is closed in $D = A \cup B$

Since $A \cap B = \phi$, $A \cup B = D$, so A & B are complements of each other in $D = A \cup B$ i.e

$A = D - B$; $B = D - A$

B closed $\Rightarrow$ A open ; A closed $\Rightarrow$ B open.

Hence both A & B is open in D.

Converse: Let A & B be both open and closed in $D = A \cup B$

$\because$ A is closed in $D = A \cup B$

$\therefore A = \bar{A}_D = \bar{A} \cap D = \bar{A} \cap (A \cup B)$

$= (\bar{A} \cap A) \cup (\bar{A} \cap B)$

$= A \cup (\bar{A} \cap B)$ $\longrightarrow (1)$

Now $A \cap (\bar{A} \cap B) = (A \cap \bar{A}) \cap B = A \cap B = \phi$

Consequently we have

$A \cap (\bar{A} \cap B) = \phi$ & $A \cup (\bar{A} \cap B) = A$

$\Rightarrow \bar{A} \cap B = \phi$

Similarly (interchanging A & B) $A \cap \bar{B} = \phi$

Hence A & B are seperated in (X, T).

Theorem # Prove that two open (closed) subsets of a top-space (X, T) are separated iff they are disjoint.

Proof # Let the sets be separated. Then since the separated sets are always disjoint, whether these are open or closed. Hence open (closed) separated sets are disjoint.

Conversely let A & B be two disjoint open sets i.e

$A \cap B = \phi$

We need to prove that

$A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$

If possible let $A \cap \bar{B} \neq \phi$ and let $p \in A \cap \bar{B}$

Then $p \in A$ & $p \in \bar{B}$

$\because$ A is an open set containing the point p which is an accumimulation point of B

$\therefore$ A must contain at least one point of B

$\Rightarrow A \cap B \neq \phi$

Which contradicts our hypothesis $A \cap B = \phi$

Thus $A \cap \bar{B} = \phi$

Similarly $\bar{A} \cap B = \phi$

Thus A & B are separated.

Let A & B be two disjoint closed sets i.e $A \cap B = \phi$

We need to prove that

$$A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi$$

$\because$ A & B are closed.

$\therefore$ $\bar{A} = A$ and $\bar{B} = B$

Thus $A \cap \bar{B} = A \cap B = \phi$

$\bar{A} \cap B = A \cap B = \phi$

Thus A & B are separated.

Theorem# Prove that if the union of two separated sets is an open (or closed) set, then the individual sets are also open (or closed).

Proof# Let A & B be two separated sets and $A \cup B$ be their union.

Then

$$A \cap \bar{B} = \phi \qquad \bar{A} \cap B = \phi$$

$\Rightarrow A \subseteq (\bar{B})^c$

$B \subseteq \bar{B} \Rightarrow (\bar{B})^c \subseteq B^c \Rightarrow B \cap (\bar{B})^c \subseteq B \cap B^c = \phi$

$\Rightarrow B \cap (\bar{B})^c = \phi \longrightarrow (1)$

(a) Let $A \cup B = G$, be an open set

$\because$ $\bar{B}$ is closed $\therefore$ $(\bar{B})^c$ is open

So $G \cap (\bar{B})^c$ must be open

$$G \cap (\bar{B})^c = (A \cup B) \cap (\bar{B})^c$$

$$= \{A \cap (\bar{B})^c\} \cup \{B \cap (\bar{B})^c\}$$

$$= A \cup \phi = A \qquad \because A \subseteq (\bar{B})^c \text{ \& } B \cap (\bar{B})^c = \phi \text{ by (1)}$$

$\Rightarrow A$ is open

Interchanging A & B we conclude that set B is also open.

(b) Let $A \cup B = F$, be a closed set

Then

$$F = \bar{F} \Rightarrow A \cup B = \overline{A \cup B} = \bar{A} \cup \bar{B}$$

$$\therefore \bar{A} = \bar{A} \cap (\bar{A} \cup \bar{B}) \qquad \because \bar{A} \subseteq \bar{A} \cup \bar{B}$$

$$= \bar{A} \cap (A \cup B)$$

$$= \{\bar{A} \cap A\} \cup \{\bar{A} \cap B\}$$

$$= A \cup \phi = A \qquad A \subseteq \bar{A}$$

$\Rightarrow \bar{A} = A$

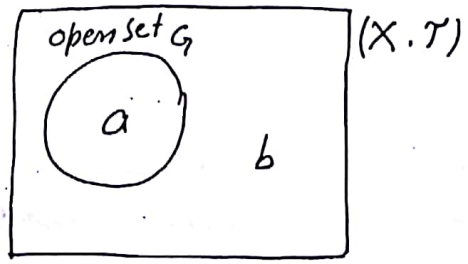
Similarly $\bar{B} = B$

. which proves that A & B are closed sets.

T_0 - Space

A top- space (X, T) is said to be T_0-space if for any two distinct points a, b of X, there exists an open set which contains one of them but not the other. OR

equivalently, for every pair of distinct points, there exists a nbd of one of these points which does not contain the other.



Examples

(1) Let X be an indiscrete space with at least two points. Then X is not a T_0- space.

(2) Let $X = \{a, b\}$ $T_2 \{\phi, \{a\}, X\}$ so that (X, T) is

the Sierpinski space. Then (X,T) is a T_0-space.
3) The real line R with usual topology is a T_0-space. 4) Every infinite set with a co-finite topology is a T_0-space.

Theorem# Subspaces of T_0-spaces are T_0-spaces.

Proof Let (X,T) be a T_0-space and $Y \subseteq X$, a subspace of X.

Let a, b be two distinct points of Y. Then $a, b \in X$.

$\because$ X is a T_0-space

$\therefore$ There is an open set G which contains one of the points but no the other.

But then

$$U = Y \cap G$$

Contains one of the points but not the other.
Hence subspace (Y, T_Y) is also a T_0-space.

Theorem# A top-space (X,T) is a T_0-space iff one of the following conditions holds

(i) For every pair of distinct points p, q in X either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$ i.e closures of distinct points are distinct

(ii) Closures of distinct points are distinct

(iii) For every point $p \in X$, $\{p\}^d$ is a union of closed sets.

Proof#(i) Let (X,T) be a T_0-space and p, q be two distinct points in X. Then there exists an open set G which contains one of the points but not the other.

Let $p \in G$ & $q \notin G$

Then the open set G containg p, has an empty

intersection with the (one-pointic) set $\{q\}$. Hence $p \notin \{q\}^d$ Also $p \neq q$ Consequently $p \notin \overline{\{q\}}$
Similarly if $q \in G$ & $p \notin G$, then $q \notin \overline{\{p\}}$
Thus either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$
Conversely, let the condition holds i.e
either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$
Let $p \notin \overline{\{q\}}$. Let $V = X - \overline{\{q\}}$. Then V is an open set such that

$p \in V$ and $q \notin V$ $\because q \notin \overline{\{q\}}$

Hence $(X\ T)$ is a T_0-space.

(ii) Let (X, T) be a T_0-space and p, q be two distinct points in X. Then by (i) above either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$
Let $p \notin \overline{\{q\}}$. Also $p \in \overline{\{p\}}$. Hence

$$\overline{\{p\}} \neq \overline{\{q\}}$$

OR

By T_0-axiom, there exists an open set G containing one of p, q but not the other.
Let $p \in G$ but $q \notin G$.
Clearly G^c is a closed set containg q but not p
Now from definition $\overline{\{q\}}$ is the intersection of all closed super sets of $\{q\}$, therefore

$$q \in \overline{\{q\}} \subseteq G^c$$

But $p \notin \overline{\{q\}}$
Hence $\overline{\{p\}} \neq \overline{\{q\}}$

Conversely suppose that $p \neq q \Rightarrow \overline{\{p\}} \neq \overline{\{q\}}$. We are to prove that (X, T) is a t_0-space i.e either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$
Let $p \in \overline{\{q\}}$ and $q \in \overline{\{p\}}$
Then $\overline{\{p\}} = \overline{\overline{\{p\}}} \supseteq \overline{\{q\}} = \overline{\overline{\{q\}}} \supseteq \overline{\{p\}}$ $\because p \in \overline{\{q\}}$
$\Rightarrow \overline{\{p\}} = \overline{\{q\}}$ $\Rightarrow \overline{\{p\}} \subseteq \overline{\{q\}}$
which is a contradiction $\Rightarrow \overline{\{p\}} \subseteq \overline{\overline{\{q\}}} = \overline{\{q\}}$
$\overline{\{q\}} \subseteq \overline{\{p\}}$

OR

$\because \quad \overline{\{p\}} \neq \overline{\{q\}}$

$\therefore$ there must exist a point z of X which is contained in one of them but not the other.

Let $z \in \overline{\{p\}}$ but $z \notin \overline{\{q\}}$

If we had $p \in \overline{\{q\}}$, we would have

$\{p\} \subseteq \{\overline{q}\}$

and $\overline{\{p\}} \subseteq \overline{\{\overline{q}\}} = \overline{\{q\}}$

and so $z \in \overline{\{p\}} \subseteq \overline{\{q\}}$

which is a contradiction.

Hence $p \notin \overline{\{q\}}$

and so $p \in \overline{\{q\}}^c$

and thus $\overline{\{q\}}^c$ is an open set containing p but not q

$\Rightarrow (X, T)$ is a T_0-space.

Example # Let (X, T_1) be an indiscrete space and (Y, T_2) a T_0-space. If $f: (X, T_1) \longrightarrow (Y, T_2)$ is continuous, then f must be constant. OR A continuous function from an indiscrete space to a T_0-space is necessarily constant.

Sol # Let f be not constant and let $f(x_1) = a$ and $f(x_2) = b$, where $a \neq b$.

As (Y, T_2) is a T_0-space, there exists an open set $V \in T_2$, such that V contains one of the points a and b. Let $a \in V$, $b \notin V$

$\because \quad x_1 \in \bar{f}^1(V)$ and $\bar{f}^1(V)$ is a proper subset of X.

As f is continuous, $\bar{f}^1(V) \in T_1$.

Hence $\bar{f}^1(V)$ must be the null set ϕ, since the topology T_1 is indiscrete. But then $x_1 \in \bar{f}^1(V) = \phi$, which is impossible. Hence f must be constant.

T_1- Space

A top-space (X, τ) is said to be a T_1-space if for every pair of distinct points a, b in X, there exist two open sets G_a and G_b such that

$a \in G_a \quad b \in G_b \quad a \notin G_b \quad b \notin G_a$

In other words, every pair of distinct points is weakly separated in (X, T)

Comments #1 It may be noted that in $[T_1]$ axiom $G_a \cap G_b$ may or may not be empty ie may or may not be disjoint.

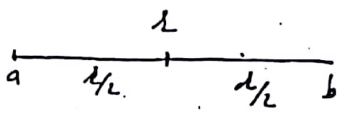
(2) Clearly by definition T_1-space is a T_0-space But converse is not true in general e.g

The Sierpinski space $X = \{a, b\}$ with $\tau = \{\phi, \{a\}, X\}$ is a T_0-space but not a T_1-space. Here every open set which contains b (there is only one such set X) also contains a

Examples #

(1) Every discrete space is a T_1-space

(2) The real line R with usual topology is a T_1-space. If $a, b \in R$, $a \neq b$ and $|a-b| = \lambda$, then $B_1(a : \lambda/2)$, $B_2(b, \lambda/2)$ are required open sets.



(3) Any infinite X with a co-finite topology is a T_1-space

For $x, y \in X$, $X - \{x\}$, $X - \{y\}$ are required open sets

(5) $X = \{a, b, c\}$ and $\tau_2 \{\phi, \{a\}, \{a, b\}, X\}$. Then (X, T) is a T_0-space but not a T_1-space.

Here all the open sets containing a, b viz $\{a, b\}$ and X also contain the point a. Thus $a \neq b$ cannot be weakly separated in (X, T).

Theorem # A T_0-space in which for any pair of distinct points a, b, $a \in \overline{\{b\}}$ implies $b \in \overline{\{a\}}$ is a T_1-space.

Proof # Let (X, T) be a topological space in which the given condition holds. Let a, b be a pair of distinct points in X. Since (X, T) is a T_0-space, either $a \notin \overline{\{b\}}$ or $b \notin \overline{\{a\}}$.

Let $a \notin \overline{\{b\}}$. Again since $b \in \overline{\{a\}} \Rightarrow a \in \overline{\{b\}}$ by given condition; therefor $b \notin \overline{\{a\}}$.

Thus

$$\because \quad a \notin \overline{\{b\}} \quad \text{and} \quad b \notin \overline{\{a\}}$$

$$\Rightarrow \quad a \in X - \overline{\{b\}} \quad \text{and} \quad b \in X - \overline{\{a\}}$$

But then $X - \overline{\{b\}}$ and $X - \overline{\{a\}}$ are open sets such that

$$a \in X - \overline{\{b\}} \quad \text{and} \quad b \in X - \overline{\{a\}}$$

and $a \notin X - \overline{\{a\}}$, $b \notin X - \overline{\{b\}}$

Thus (X, T) is a T_1-space.

Theorem # A top-space (X, T) is a T_1-space iff every one pointic set is closed set.

Proof Let (X, T) be a T_1-space and $\{x\}$ a one pointic subset of X. Let y be any point in X other than x. Then by axiom T_1, there exists an open set V such that

$$y \in V \qquad x \notin V$$

$$\Rightarrow \quad V \cap \{x\} = \phi$$

$\Rightarrow$ y can not be an accumulation point of the set $\{x\}$ i.e $\{x\}$ contains all of its limit points.

$\Rightarrow$ $\{x\}$ is closed

OR

We show that $\{x\}^c$ is open.

Let y be any point in X other than x. By T_1-axiom $\exists$ an open G_y such that

$y \in G_y$ and $x \notin G_y$

Also $G_y \cap \{x\} = \phi$

$\Rightarrow G_y \subseteq \{x\}^c$

$\Rightarrow y$ is an interior point of $\{x\}^c$

Since y is an arbitrary, every element of $\{x\}^c$ is an interior point of $\{x\}^c$. So $\{x\}^c$ is open and $\{x\}$ is closed or

$$\{x\}^c = \cup\{y : y \neq x\} \subseteq \cup\{G_y : y \neq x\} \subseteq \{x\}^c$$

$\Rightarrow \{x\}^c = \cup\{G_y : y \neq x\}$

$\Rightarrow \{x\}^c$ is open

$\Rightarrow \{x\}$ is closed.

Conversely let (X, T) be a top-space in which every one pointic set is closed. Let a, b be two distinct points in X. Then $\{a\}$ and $\{b\}$ are closed.

$\Rightarrow X - \{a\}$ and $X - \{b\}$ are open sets

Now $b \in X - \{a\}$ $\quad b \notin X - \{b\}$

$a \in X - \{b\}$ $\quad a \notin X - \{a\}$

Thus (X, T) is a T_1-space.

Example # Every metric space is a T_1-space.

Sol Let (X, d) be a metric space and a, b be two distinct elements of X and

$$d(a, b) = \alpha$$

Consider

$G_a = B(a, \alpha/2)$

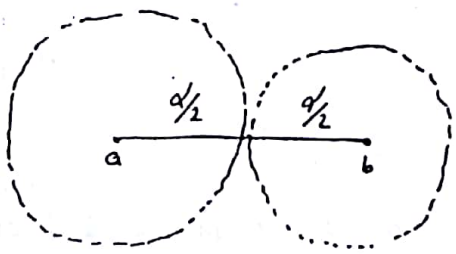
$G_b = B(b, \alpha/2)$

Then

$a \in G_a$ $\quad b \in G_b$

and $a \notin G_b$ $\quad b \notin G_a$

$\Rightarrow (X, d)$ is a T_1-space.



Theorem # The following statements are equivalent for a topological space (X, T). Also (a), (b), (c), (d) are equivalent and (a), (b), (e) equivalent.

(a) (X, T) is a T_1-space

(b) Every one pointic set is closed in X

(c) Every finite set is closed in X

(d) For every point $x \in X$, $\{x\}^d = \emptyset$

(e) For each subset A of X, A is the intersection of its open super sets

(f) For every subset $A \subseteq X$, any accumulation point point p of A (i.e every open set, containing p, intersects A in a denumerably infinite collection of points).

Proof # (a) $\Rightarrow$ (b)

Suppose that X is a T_1-space. Let $x \in X$ Consider the set $X - \{x\} = \{x\}^c$. We shall that $\{x\}^c$ is open. Let $y \in X - \{x\} = \{x\}^c$. Then $y \neq x$. Since X is a T_1-space, there is an open set U of V such that

$x \in U \quad y \notin U$

$y \in V \quad x \notin V \quad \Rightarrow \{x\} \cap V = \emptyset$

so $y \in V \subseteq \{x\}^c \quad \Rightarrow V \subseteq \{x\}^c$

$\Rightarrow$ y is an interior point of $\{x\}^c$

Since y is an arbitrary element of $\{x\}^c$, $\{x\}^c$ is open and hence $\{x\}$ is closed. Thus every one pointic set in X is closed.

(b) $\Rightarrow$ (c)

Suppose that each one pointic subset of X is closed and A be any finite subset of X. Then

$$A = \bigcup_{x \in A} \{x\}$$

$\because$ each one pointic subset in X is closed

$\therefore$ A being the union of finite collection of closed subsets is closed.

(c) $\Rightarrow$ (d)

Let every finite set is closed in (X, T).
Then for every $x \in X$, $\{x\}$ is closed.
Hence $\{x\} = \overline{\{x\}} = \{x\} \cup \{x\}^d$
$\Rightarrow \{x\}^d \subseteq \{x\}$
But $x \notin \{x\}^d$ because any open set containg x will not intersect $\{x\}$ in a point other than x
Hence $\{x\}^d = \phi$ $\because$ if $\{x\}^d \neq \phi$, $\{x\}^d \not\subseteq \{x\}$ because $x \notin \{x\}^d$ and then a pot different from x will be in $\{x\}^d$ wich would not be $\{x\}$
Conversely let $\{x\}^d = \phi$
Then $\overline{\{x\}} = \{x\} \cup \{x\}^d = \{x\}$
$\Rightarrow \{x\}$ is closed
Thus (d) $\Rightarrow$ (b) $\Rightarrow$ (a)
(d) $\Rightarrow$ (e)

Proof # (e) Let (X, T) be a T_1- space. Then (b) holds and set $\{x\}$ is closed.
Hence for any point $b \notin A$, there is an open set $X - \{b\}$ such that

$$A \subseteq X - \{b\} \quad \text{and} \quad b \notin X - \{b\}$$

Hence $A = \cap \{X - \{b\} : b \in X - A\}$
$\Rightarrow$ Intersection of all open sets, containing A, is the set A.
Conversely let A be equal to the intersection of all open sets containing A.
Let a, b be a pair of distinct points in X.
By taking $A = \{a\}$, it follows that the intersection of all open sets containg the point a, is the set $\{a\}$.
Hence there exists an open set U such that $a \in U$ and $b \notin U$. Similarly by taking $A = \{b\}$, there exists an open set V such that $b \in V$, $a \notin V$.
Hence (X, T) is a T_1- space

Equivalence of (a) & (f)
Let (X, T) be a T_1- space and let p be an

accumulation point of a subset A in $(X, \mathcal{T})$. Let U be an open set containg p. Then U intersects A in some point $x_1 \neq p$. By (b) $\{x_1\}$ is a closed set in $(X, \mathcal{T})$. Hence $U - \{x_1\} = U \cap \{X - \{x_1\}\}$ is an open set containg p and it intersects A is some point $x_2 \neq p$. Also $x_2 \neq x_1$. Now $\{x_1, x_2\}$ is a closed set by (c). Hence $U - \{x_1, x_2\}$ is an open set containg p and it must intersect A in some point $x_3 \neq p$. Also $x_3 \neq x_1$, $x_3 \neq x_2$. Continuing this process, we obtain an infinite sequence of distinct points $x_1, x_2, x_3 \ldots$ in $U \cap A$. Consequently p is an ω-accumulation point of A.

Conversely let any accumdation point of a subset A of X be an ω-accumulation point of A. Let (X, T_1) be not a T_1-space. Then by (b) some one pointic set must be not closed in $(X, \mathcal{T})$. Let Let $\{x\}$ be not closed i.e $\{x\} \neq \overline{\{x\}}$ and let $y \in \overline{\{x\}}$, where $y \neq x$ Then y is an accumulation point of $\{x\}$ but it cannot be an ω-accumulation point of $\{x\}$, since $\{x\}$ consist of only one point. Thus if $(X, \mathcal{T})$ is not T_1 space, an accumulation point need not be an ω-accumulation point. Thus $(X, \mathcal{T})$ is a T_1-space.

Corollary # A space X is a T_1-space iff every finite subset of X is closed. Proved above in (c)

Corollary # Every finite T_1-space is discrete.

Proof # Let X be a finite T_1-space. Then every subset of X is closed and so every subset of X, being the complement of closed set is open. Hence X is discrete.

Theorem # In a T_1-space X, A point x is a limit point of a set A iff every open set containing x contains an infinite number of distinct points of A.

Proof # Let every open set containing x contains an infinite number of points of A. Then x is a limit point of A.

Conversely let x be a limit point of A. Suppose that there is an open set G containing x for which $A \cap G$ is finite. Let $G \cap A \setminus \{x\} = \{x_1, x_2, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\} = B$

Then B being a finite subset of a T_1-space is closed.

Now $x \notin B$

$\Rightarrow x \in B^c$, B^c is an open set

Now $x \in B^c \cap G$, $B^c \cap G$ is an open set

and $(B^c \cap G) \cap A \setminus \{x\} = B^c \cap \{G \cap A \setminus \{x\}\}$

$= B^c \cap B = \phi$

Thus x would not be a limit point of A which is a contradiction. Hence $A \cap G$ is infinite for every open set G containing x.

Corollary # In a T_1-space, no finite subset has a limit point.

Proof # Let X be a T_1-space, A a finite subset of X. Let x be a limit point of A. Then every open set containing x will contain an infinite number of points of A which is impossible because A itself is finite. Hence no point of X is a limit point of A.

Theorem # (i) The coarsest T_1-topology, that can be defined on a set X, is the co-finite topology on X

(ii) The intersection of any collection of T_1-tops on a set X is also a T_1-topology.

Proof # (i) Let τ be co-finite topology on X. Then τ is a T_1-topology because finite sets are closed in (X, τ). Again, for any T_1-topology τ' on X, every finite set is τ'-closed, so that complements of finite sets are τ'-open

Consequently $\tau \subseteq \tau'$. Hence τ is the coarsest among the T_1- topologies that can be defined on X

(ii) Let $\{\tau_\alpha\}$ be a given collection of T_1- topologies on X and τ_1 be their intersection. Then τ_1 is aloso a topology on X. By (i) τ_1 also contains the co-finite topology τ on X i.e τ_1 is finer than τ. As τ is a T_1- topology on X, τ_1 is also a T_1- topology.

Theorem # If (X, τ) is a T_i - space, τ' is a topology on X, finer than τ, then (X, τ') is also a T_i - space, for i=0, 1 and 2.

Proof # As τ' is finer than the topology τ, every τ-open set is also also a τ'- open set. Let p, q be any two distinct points in X.

If G is a τ-open set containing one of p, q but not the other, then G is τ'-open set with the same property. Hence if (X, T_0) is T_0 space, then (X, τ') is also T_0 space

If U and V are two τ- open sets, which separate p and q weakly (respectively strongly), then U and V are two τ'- open sets, which separate p and q weakly (respectively strongly). Hence if (X, τ) is a T_1- space or a T_2 - space, then (X, τ') is also respectively a T_1- space or a T_2 - space.

T_2 - Space # (Hausdorff space)

A top- space (X, τ) is said to be T_2- space iff for every pair of distinct points a, b in X, there exist two open sets G & H such that

$a \in G$, $b \in H$ & $G \cap H = \phi$

i.e every pair of distinct points is strongly separated.

or equivalently, distinct points have distinct nbds.

Examples #

(1) Every metric space is a Hausdorff space

Proof Let (X, d) be a metric space and a, b be any two distinct points of X. Then $d(a,b) = r > 0$ and $B(a, \frac{r}{2})$ $B(b, \frac{r}{2})$ are disjoint open sets containing a and b respectively

(2) Every discrete space is a Hausdorff space.

(3) No indiscrete space with more than one point is a Hausdorff space.

(4) No infinite set with a Co-finite topology is Hausdorff

Let X be an infinite set with a co finite topology. Then open sets in X are ϕ and all those sets whose complements are finite. This space is T_1-space but no T_2-space. We note that for this top-space non of the two open sets are disjoint because

U open $\Rightarrow$ U is infinite

V open $\Rightarrow$ V is " "

$\Rightarrow X-U$ & $X-V$ are finite

$\Rightarrow (X-U) \cup (X-V)$ is finite

$\Rightarrow U^c \cup V^c$ is finite

$\Rightarrow (U \cap V)^c$ is finite

$\Rightarrow U \cap V$ is infinite

$\Rightarrow U \cap V \neq \phi$

$\Rightarrow (X, T)$ can not be Hausdorff because $U \cap V = \phi$ is a requirement of a Hausdorff space.

OR

Suppose that X is a T_2-space. Then for any two distinct points a, b of X there are open sets U and V such that

$\therefore a \in U, b \in V$ and $U \cap V = \phi$.

But the complements of open sets are finite. So that from

$$U \cap V = \phi$$

we have $(U \cap V)^c = \phi^c$

$$U^c \cup V^c = X$$

So X is a finite set, a contradiction. Hence X is not T_2-space.

Comments # Every T_2-space is obviously T_1-space but the converse is not true e.g.

Let X be an uncountabe set and T the topology of cocountable complements on X. Then (X, T) is a T_1-space since every finite subset (being countable) is closed in (X, T). But it is not a T_2-space, since no two (non-null) (disjoint) open sets in (X, T) are disjoint.

In fact if G and H are be two non-empty disjoint open sets in (X, T), then

$$(X - G) \cup (X - H) = X - (G \cap H) = X - \phi = X$$

which is impossible because each of the closed sets $X - G$ and $X - H$ being cocountable $(X - G) \cup (X - H)$ is also cocountable, whereas the set X is uncountable.

Theorem # Every subspace of a Hausdorff is Hausdorff OR The property of being a T_2-space is hereditary

Proof # Let (A, T_A) be a subspace of a Hausdorff space (X, T). Let a, b be two distinct points of A. Then $a, b \in X$

$\because$ X is a T_2-space

$\therefore$ $\exists$ open sets U & V in X such that $a \in U$, $b \in V$ and $U \cap V = \phi$

Now $U_A = U \cap A$ and $V_A = V \cap A$ are two open sets

in A such that

$a \in U_A$, $b \in V_A$ & $U_A \cap V_A = (U \cap A) \cap (V \cap A)$

$= (U \cap V) \cap A$

$= \phi \cap A = \phi$

which shows that $(A, \mathcal{T}_A)$ is a Hausdorff space.

Diagonal of a Top-Space

Let $(X, \mathcal{T})$ be a top-space. Then

$D = \{(x, x) \mid x \in X\}$ is called the diagonal of $X \times X$

Theorem # A top-space $(X, \mathcal{T})$ is Hausdorff iff the diagonal D of $X \times X$ is closed in $X \times X$

Proof # Let $(X, \mathcal{T})$ be a Hausdorff space. We show that D is closed in $X \times X$ by proving that D^c is open in $X \times X$

Let $(a, b) \in D^c \Rightarrow a \neq b$, $a, b \in X$

$\because$ X is a Hausdorff space

$\therefore \exists$ open sets U & V such that

$a \in U$, $b \in V$ and $V \cap U = \phi$

$\Rightarrow (a, b) \in U \times V$ where $U \times V$ is open in $X \times X$

and $U \times V \subseteq D^c$ because $U \cap V = \phi$

$\Rightarrow (a, b) \in U \times V \subseteq D^c$

$\Rightarrow D^c$ is open $\Rightarrow D$ is closed in $X \times X$

Conversely Let D be closed in $X \times X$ in

Consider $a, b \in X$ such that $a \neq b$

$\Rightarrow (a, b) \in D^c$ where D^c is open in $X \times X$

and

$D^c = \bigcup (U_\alpha \times V_\alpha)$ U_α, V_α are open in X

$\Rightarrow (a, b) \in U_\alpha \times V_\alpha$ for some $\alpha \in I$

$\Rightarrow a \in U_\alpha , b \in V_\alpha$ and $U_\alpha \times V_\alpha \subseteq D^c$.

$\Rightarrow U_\alpha \cap V_\alpha = \emptyset$

$\Rightarrow (X, T)$ is a Hausdorff space.

Theorem # The following statements about a top-space are equivalent

(a) # X is a Hausdorff space

(b) # The diagonal $D = \{(x, x) : x \in X\}$ is closed in $X \times X$

Proof # (a) $\Rightarrow$ (b)

Suppose that X is a Hausdorff space and Let $D = \{(x,x) : x \in X\}$ be the diagonal of $X \times X$. We prove D is closed by proving that D^c is open.

Let $(a,b) \in D^c$. Then $a \neq b$, $a, b \in X$

$\because$ X is a Hausdorff space

$\therefore \exists$ open sets U and V such that

$a \in U \quad b \in V$ and $U \cap V = \emptyset$

But $U \cap V = \emptyset \Rightarrow U \cap V \subseteq D^c$

So $(a,b) \in U \times V \subseteq D^c$ where $U \times V$ are open

$\Rightarrow D^c$ is open in $X \times X$ in D.

$\Rightarrow D$ is closed in $X \times X$

(b) $\Rightarrow$ (a)

Suppose that the diagonal D of $X \times X$ is closed in $X \times X$. Let $a, b \in X$, $a \neq b$. Then

$(a,b) \notin D \quad \because a \neq b$

$\Rightarrow (a,b) \in D^c$

Since D^c is open in $X \times X$, there are open sets U and V in X such that

$(a,b) \in U \times V \subseteq D^c$

$\Rightarrow a \in U , b \in V$ and $U \cap V = \emptyset \quad \because U \times V \subseteq D^c$

Thus X is a Hausdorff space.

Theorem # A top-space $(X, \mathcal{T})$ is a Hausdorff space iff any one of the conditions holds.

(i) For every point $x \in X$, $\{x\} = \cap\{\overline{N_x} : N_x \in \mathcal{N}_x\}$ where $\mathcal{N}_x$ denotes the neighbourhood filter of the point x

(ii) The diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed set in the product space $(X \times X, \mathcal{T} \times \mathcal{T})$

Proof # Let $(X, \mathcal{T})$ be a Hausdorff space and let $x \in X$. Then for any point $y \neq x$, there exist two disjoint nbds N_x and N_y of x and y respectively such that

$$N_x \cap N_y = \phi$$
$$\Rightarrow N_x \subseteq X - N_y = N_y^c$$
$$\Rightarrow \overline{N_x} \subseteq \overline{X - N_y} \subseteq \overline{X - V} = X - V$$

where V is an open set satisfying $y \in V \subseteq N_y$

So we get

$$\overline{N_x} \cap V = \phi$$
$$\Rightarrow y \notin \overline{N_x} = \overline{N_x}$$

Hence $y \notin \cap\{\overline{N_x} : N_x \in \mathcal{N}_x\}$

$$\Rightarrow \cap\{\overline{N_x} : N_x \in \mathcal{N}_x\} = \{x\}$$

Conversely let the condition (i) be true in $(X, \mathcal{T})$. Then for any $y \neq x$, there exists N_x such that $y \notin \overline{N_x}$

Then $y \in X - \overline{N_x}$ where $X - \overline{N_x}$ is open

$\Rightarrow X - \overline{N_x}$ is an open nbd of y

Thus $x \in N_x$ $\quad y \in X - \overline{N_x}$ and $N_x \cap (X - \overline{N_x}) = \phi$

$\Rightarrow x$ & y are strongly separated by two disjoint nbds.

$\Rightarrow (X, \mathcal{T})$ is a Hausdorff space.

(ii) Let $(X, \mathcal{T})$ be a Hausdorff space and (Z, δ) denote the product space $(X \times X, \mathcal{T} \times \mathcal{T})$

Let $(x, y) \in Z - \Delta = \Delta^c$

Then $x \neq y$

$\because \{x\} = \cap\{\overline{N_x} : N_x \in \mathcal{N}_x\}$ by (i)

$\therefore$ there exists a nbd N_x of x such that $y \notin \overline{N_x}$

Then $X - \bar{N}_x$ is an open nbd of y and since N_x and $X - \bar{N}_x$, are disjoint, $N_x \times (X - \bar{N}_x) \subseteq \Delta^c$

Also $(x, y) \in N_x \times (X - \bar{N}_x) \subseteq \Delta^c$

$\Rightarrow$ Δ^c is an open set

$\Rightarrow$ Δ is a closed set in product space (Z, σ)

Conversely let the condition (ii) hold in (X, T). Let x, y be two distinct points in X. Then

$$(x, y) \in Z - \Delta = \Delta^c$$

$\because$ Δ^c is an open set in (Z, σ) and $\{ N_x \times N_y : N_x \in \mathcal{N}_x, N_y \in \mathcal{N}_y \}$ forms a base of the product topology σ, there exists a suitable $N_x \times N_y$ such that

$$(x, y) \in N_x \times N_y \subseteq \Delta^c$$

$$\Rightarrow \quad N_x \cap N_y = \phi$$

Consequently x and y are separated strongly by the disjoint nbds N_x and N_y of x, y respectively. Hence (X, T) is a Hausdorff space.

Corollary # Let f and g be two continuous functions on a topological space (X, T), with values in a Hausdorff space (X, T'). Then

(1) $F = \{ x \in X : f(x) = g(x) \}$ is a closed subset in (X, T)

(2) If $f(x) = g(x)$ for all points x in a subset A, which is everywhere dense in X, then f and g are identical.

Proof # (1) Let h be a mapping of (X, T) into $(Y, T') \times (Y, T')$ defined by

$$h(x) = (f(x), g(x))$$

Then since f and g are continuous, therefor h is continuous. Let F be inverse image of Δ of $Y \times Y$ under h. As Δ is closed in $Y \times Y$, F is closed set in (X, T)

(2) Let $F = \{ x \in X : f(x) = g(x) \}$. Then

$A \subseteq F$

$\Rightarrow \quad \bar{A} \subseteq \bar{F}$

$\because$ A is dense in everywhere

$\therefore \quad \bar{A} = X$

$\Rightarrow \quad X \subseteq \bar{F}$

Also by (i) F is a closed set, so that $\bar{F} = F$

$\Rightarrow \quad X \subseteq \bar{F} = F$

But $F \subseteq X$

$\Rightarrow \quad F = X$

Thus $f(x) = g(x)$ holds for all points $x \in X$: hence f and g are identical.

Theorem ≈ A top-space (X, τ) is a Hausdorff space iff for any two distinct points a, b in X, there are closed set F_1 and F_2 such that

(i) $X = F_1 \cup F_2$

(ii) $a \in F_1 \quad b \notin F_1 \quad b \in F_2, a \notin F_2$

Proof# Let $(X \tau)$ be Hausdorff and a, b be two distinct points of X. Then $\exists$ open sets U and V such that

$a \in U \quad b \in V \quad$ and $U \cap V = \phi$

But then

$U \cap V = \phi \Rightarrow (U \cap V)^c = \phi^c$

$\Rightarrow U^c \cup V^c = X$

Now $a \notin V \Rightarrow a \in V^c = F_1$ closed set in X

and $b \notin U \Rightarrow b \in U^c = F_2$ closed set in X

as required

Conversely suppose that for any two distinct points a, b in X $\exists$ closed sets F_1 and F_2 such that

$X = F_1 \cup F_2 \quad a \in F_1 \quad b \notin F_1$ and $b \in F_2, a \notin F_2$

Then F_1^c and F_2^c are open sets and.

$b \in F_1^c$ and $a \in F_2^c, \quad a \notin F_1^c \quad b \notin F_2^c$

and $X^c = (F_1 \cup F_2)^c \Rightarrow \phi = F_1^c \cap F_2^c$

Hence X is Hausdorff.

Theorem # A top-space (X, τ) is Hausdorff iff for every point a of X

$$\{a\} = \bigcap_{\alpha} C_\alpha$$

where each C_α is a closed set containing an open set U which contains a

i.e C_α is a closed neighbourhood of a

Proof # Suppose that X is a Hausdorff space and a, is any point of X. For any point $b \neq a$, there exist open sets U and V such that

$$a \in U, b \in V \text{ and } U \cap V = \phi$$

$$\Rightarrow U \subseteq V^c$$

$\because$ V^c is closed and $U \subseteq V^c$

$\therefore$ $V^c = C_\alpha$ for some α i.e V^c is a closed nbd of a

Also $b \in V \Rightarrow b \notin V^c = C_\alpha$

So b is not in at least one closed nbd of a.

Since b, is an arbitrary point of X different from a, $b \notin \bigcap_{\alpha} C_\alpha$, C_α a closed nbd of a, so a is the only point which is in every nbd of a. Thus

$$\{a\} = \bigcap_{\alpha} C_\alpha$$

Conversely suppose that for a top-space X and a point a of X

$$\{a\} = \bigcap_{\alpha} C_\alpha$$

where $\{C_\alpha\}$ is the family all closed nbds of a.

Let b be any point different from a. Then

$b \notin \bigcap C_\alpha$

$\Rightarrow$ There is a closed nbd C, one of C_α which does not contain b

Then $b \in C^c$ where C^c is open

Also there is an open set U containing a, s.t

$a \in U \subseteq C$

clearly $U \cap C^c = \phi$, $a \in U$, $b \in C^c$

$\Rightarrow X$ is a Hausdorff space.

Theorem # An infinite Hausdorff space X contains an infinite sequence of non-empty, disjoint, open sets

Proof # If X has no limit points, then X must have the discrete topology and any infinite sequence of distinct points of X would serve as the desired sequence.

Suppose that x is a limit point of X. Choose x_1 to be any point of X different from x. By $[T_2]$, there exist two disjoint open sets G_1 and V_1 such that $x_1 \in G_1$ and $x \in V_1$.

Since x is a limit point of X belonging to the open set V_1, there exist some point $x_2 \in X \cap V_1 \setminus \{x\}$. By $[T_2]$, there exist two disjoint open sets G_2^* and V_2^* such that

$$x_2 \in G_2^* \quad \text{and} \quad x \in V_2^*.$$

If we let $G_2 = G_2^* \cap V_1$ and $V_2 = V_2^* \cap V_1$, then G_2 and V_2 are two disjoint open sets contained in V_1 (and hence disjoint from G_1) containing x_2 and x, respectively.

Suppose we have already defined the points $\{x_k\}$ and the open sets $\{G_k\}$ and $\{V_k\}$ with properties

$$x_k \in G_k \subseteq V_{k-1}, \quad x \in V_k \subseteq V_{k-1} \quad \text{and}$$
$$G_k \cap V_k = \phi \quad \text{for all } k \leq n$$

Now x is a limit point of X belonging open set V_n, and so there exists some point $x_{n+1} \in X \cap V_n \setminus \{x\}$. By $[T_2]$ there exist two disjoint open sets G_{n+1}^* and V_{n+1}^* such that

$$x_{n+1} \in G_{n+1}^* \quad \text{and} \quad x \in V_{n+1}^*.$$

If we let $G_{n+1} = G_{n+1}^* \cap V_n$ and $V_{n+1} = V_{n+1}^* \cap V_n$. Then G_{n+1} and V_{n+1} are two disjoint open set contained in V_n (and hence disjoint from G_n) containing x_{n+1} and x respectively. Since the sets $\{V_n\}$ are monotone decreasing, we see that G_{n+1} is not only disjoint from G_n but is also disjoint from every G_k for $k \leq n$. Since $x_n \in G_n$,

, the infinite sequence $\{G_{in}\}$ (of nos), defined by induction, is the desired sequence of non-empty, disjoint, open sets.

Theorem # Any convergent sequence in a Hausdorff space possess a unique limit.

Proof # Let $\{x_n\}$ be a convergent sequence, having a point p as a limit point, in a Hausdorff space (X, τ). Let q be any point in X different from p. Then $\exists$ two open sets U and V such that

$$p \in U, \; q \in V \text{ and } U \cap V = \emptyset.$$

Now since p is a limit point of the sequence $\{x_n\}$ and open set U contains it, therefore $\exists$ a +ve integer m such that

$$x_n \in U \qquad \forall n \geq m$$

Then, as V is disjoint with U, V can contain at most $m-1$ elements viz $x_1, x_2, \ldots, x_{m-1}$ of the sequence $\{x_n\}$. Consequently the point q can not be a limit point of the sequence $\{x_n\}$. Hence p is the unique limit of the sequence $\{x_n\}$ in (X, τ).

OR

Suppose that X is a Hausdorff space and $\{x_n\}$ is a sequence in X, which converges to two distinct points $x \neq y$. Then $\exists$ two open sets $U \neq V$ such that

$$x \in U, \; y \in V, \; U \cap V = \emptyset$$

$\because x_n \longrightarrow x$

$\therefore \exists$ a natural number n_1 such that

$$x_n \in U \qquad \forall n \geq n_1$$

Again since $x_n \longrightarrow y$, therefore $\exists$ a natural number n_2 such that

$$x_n \in V \qquad \forall n \geq n_2$$

Let $n_0 = \max(n_1, n_2)$. Then

$x_n \in U \quad \forall n \geq n_0$

and $x_n \in V \quad \forall n \geq n_0$

$\Rightarrow x_n \in U \cap V = \phi \quad \forall n \geq n_0$

which is a contradiction. Hence $x = y$ i.e Limit is unique

Remark # Open sets which contain Limit point of a sequence must contain almost all the terms of the sequence.

The converse of the the above theorem is not true as can be seen from the example.

Example # Let X be an uncountable set and $\mathcal{T}$ be the topology of countable complements on X.

Then (X, T) is a T_1-space but not a T_2-space.

Let $\{x_n\}$ be a convergent sequence in (X, T), having limit $p \in X$. Let q be any other point in X.

As (X, T) is a T_1-space, there exists an open set U containg p but not q. This implies $\exists$ a natural no m such that

$a_n \in U \quad \forall n \geq m$

Let $V = X - \{x_{m+j} : j = 0, 1, 2 \cdots\}$. Then V (being the complement of a countable set) is an open set, containg the point q. But there does not exist any +ve integer k, such that $x_n \in V \quad \forall n \geq k$. Hence q can not be a limit point of the sequence, the sequence $\{x_n\}$ has a unique limit viz p.

Thus every convergent sequence has a unique limit in the space (X, T) although (X, T) is not a Hausdorff space.

Graph of Function # Let $f: X \to Y$ be a function. Then the graph of f is a subset of $X \times Y$ such that

$$\{(x, y) : y = f(x)\}$$ and is denoted by f

Theorem # Let $f: X \to Y$ be a (conti) function from a top-space X to a Hausdorff space Y. Then the graph

$$f = \{(x, y) : y = f(x)\} \subseteq X \times Y$$

is closed in $X \times Y$

Proof # To prove that f is closed in $X \times Y$, we show that the complement f' of f is open in $X \times Y$. To prove that f' is open we prove that it is a nbd of each of its points.

$\therefore$ Let $(a, b) \in f'$ where $f' = X \times Y - f$

$\Rightarrow b \neq f(a)$ and $b, f(a)$ are distinct points of Y.

$\because$ Y is a Hausdorff space

$\therefore$ $\exists$ open sets U & V such that

$f(a) \in U$, $b \in V$ & $U \cap V = \emptyset$

Now $W = f^{-1}(U)$ is open in X ($\because$ f is continuous) and contains a. Also V is open in Y.

$\Rightarrow$ $W \times V$ is an open set in $X \times Y$ and $(a, b) \in W \times V$ because $a \in W$, $b \in V$

Now let $(c, d) \in W \times V$

$\Rightarrow f(c) \neq d$ ($\because U \cap V = \phi$)

$\Rightarrow (c, d) \notin f$,

$\Rightarrow (c, d) \in f'$

$\Rightarrow W \times V \subseteq f'$

$\Rightarrow (a, b) \in W \times V \subseteq f'$

$\Rightarrow f'$ is nbd of each of its points and hence is open

$\Rightarrow f$ is closed in $X \times Y$

$c \in W = f^{-1}(U)$
$\Rightarrow f(c) \in U$
If $f(c) = d$
Then $f(c) \in V$
and $U \cap V \neq \phi$
Therefor $f(c) \neq d$

Theorem# Let X, Y be topological spaces and Y a Hausdorff space. Let f and g be continuous functions from X to Y. Then

$$A = \{x \in X : f(x) = g(x)\} \text{ is closed in } X.$$

i.e A subsets of X at each point of which f & g coincide is a closed set in X.

Proof# A is closed in $X \iff A^c$ is open in X.

Let $a \in A^c$. Then $f(a) \neq g(a)$

$\because f(a), g(a) \in Y$ & Y is a Hausdorff space

$\therefore \exists$ open sets U & V such that

$f(a) \in U$, $g(a) \in V$ & $U \cap V = \phi$

$\Rightarrow a \in f^{-1}(U)$, $a \in g^{-1}(V)$

$\because A \cap A^c = \phi$ and $a \in A^c$

$\therefore f^{-1}(U) \subseteq A^c$, $g^{-1}(V) \subseteq A^c$ & $f^{-1}(U) \cap g^{-1}(V) \subseteq A^c$

Since f is continuous, therefore $f^{-1}(U)$, $g^{-1}(V)$ are open in X

$\Rightarrow f^{-1}(U) \cap g^{-1}(V)$ is open in X

Now $a \in f^{-1}(U) \cap g^{-1}(V) \subseteq A^c$

$\Rightarrow$ each point of A^c is an interior point of A^c

$\Rightarrow A^c$ is an open set in X

$\Rightarrow A$ is closed set in X

Theorem# Let $f: X \to Y$, $g: X \to Y$ be continuous functions from a space X to a Hausdorff space Y and let $f(x) = g(x)$ $\forall x$ in a dense subset D of X

Then $f(x) = g(x)$ $\forall x \in X$

i.e If f, g coincide at a dense subset of X, then they coincide at X

Proof# Suppose that for some $z \in X$, we have

$$f(z) \neq g(z)$$

i.e $f(z)$, $g(z)$ are two distinct points in Y

$\because Y$ is a Hausdorff space

$\therefore \exists$ open sets U & V such that

$f(z) \in U$, $g(z) \in V$ & $U \cap V = \phi$

$\Rightarrow z \in \underbrace{f^{-1}(U)}_{=U_1}, z \in \underbrace{g^{-1}(V)}_{=V_1}$

$\because$ f, g are continuous

$\therefore$ U_1 & V_1 are open in X

$\Rightarrow$ $U_1 \cap V_1$ is an open set in X containing $z \in X$

$\because$ D is dense in X $\quad\therefore\quad \bar{D} = X$

$\Rightarrow z \in \bar{D} = D \cup D^d$

Also $z \notin D$ because $f(z) \neq g(z)$

$\Rightarrow z$ is a limit point of D

Now $U_1 \cap V_1$ is an open set containing limit point of D

Therefore

$(U_1 \cap V_1) \cap D \setminus z \neq \phi$

$\Rightarrow (U_1 \cap V_1) \cap D \neq \phi \quad \because z \notin D$

Let $c \in (U_1 \cap V_1) \cap D$

$\Rightarrow c \in U_1 \cap V_1 \Rightarrow c \in U_1, c \in V_1$

$\Rightarrow c \in f^{-1}(U), \; c \in g^{-1}(V)$

$\Rightarrow f(c) \in U \quad g(c) \in V$

$\because c \in D$

$\therefore f(c) = g(c)$

$\Rightarrow U \cap V \neq \phi$

which is a contradiction

Hence $f(x) = g(x) \quad \forall x \in X$

Def A top-space said to be 1st Countable space if $\exists$ a Countable local base at every point of X

Def A top-space is called 2nd Countable space if $\exists$ a Countable base for the topology on X

Theorem # A first Countable space X is a Hausdorff space iff every convergent sequence has a unique limit.

Proof # Let X be a Hausdorff space and $\{x_n\}$ be sequence in X. Let $\{x_n\}$ converges to x, y, $x \neq y$

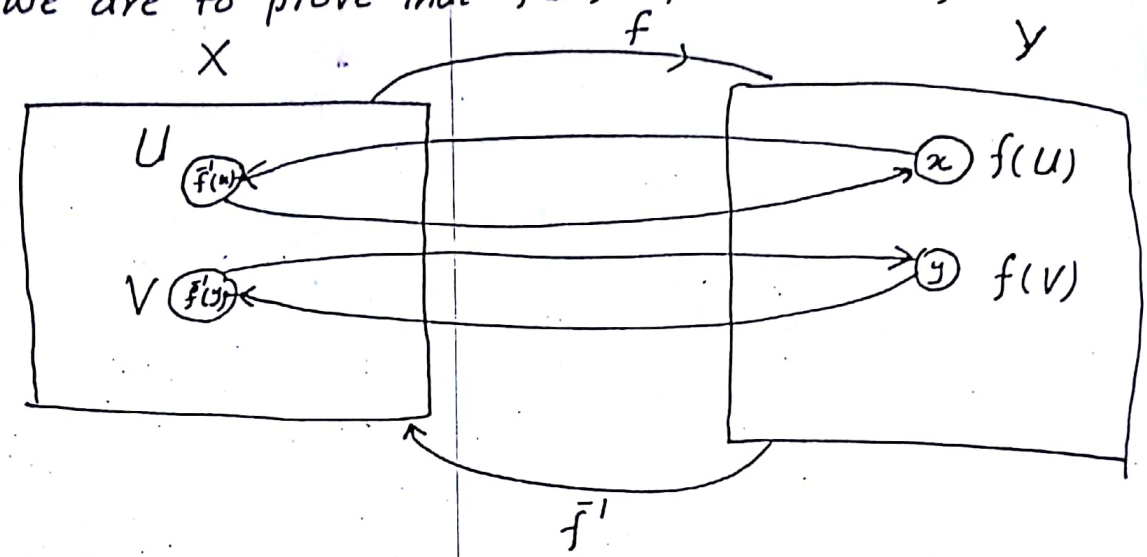
x
// Then $\exists$ natural numbers n_1, n_2 such that
$x_1 \not\parallel x$
$\because$ X is a Hausdorff space & x, y are distinct points in X
$\therefore \exists$ open sets U & V such that
$x \in U$. $y \in V$ & $U \cap V = \phi$
$\because$ x, y are limits of $\{x_n\}$
$\therefore \exists$ natural numbers n_1 & n_2 such that
$x_n \in U$ $\forall n \geq n_1$
$x_n \in V$ $\forall n \geq n_2$
Let $n_0 = \max(n_1, n_2)$
Then $x_n \in U$ $\forall n \geq n_0$
and $x_n \in V$ $\forall n \geq n_0$
$\Rightarrow U \cap V \neq \phi$
which is a Contratiction. Hence limit is unique.
Conversely suppose that every convergent sequence has a unique limit. We show that X is Hausdorff.
Let X be not Hausdorff. Then for a pair a, b of distinct points in X, every open set which contains a has a non-empty intersection with every open set which contains b.
Let $\{U_n\}$, $\{V_n\}$ be countable nested local bases at a. Then $U_n \cap V_n \neq \phi$ $\forall n$
Choose points $a_n \in U_n \cap V_n$, $n = 1, 2, \ldots\ldots$
Then $\{a_n\}$ converges to both a & b, a contradiction. Hence X is Hausdorff.

Theorem # The Homeomorphic image of a Hausdorff space is Hausdorff. OR To be a Hausdorff is a topological property.

Proof # Let f be a homeomorphism from a Hausdorff space X onto a space Y. Then f is bijective, bicontinuous and open.

Now $f(X) = Y$ $\because$ f is onto

We are to prove that $f(X) = Y$ is Hausdorff.



Let x, y be a pair of non-equal points of Y

Then $f^{-1}(x), f^{-1}(y)$ is a pair of distinct points in X because f^{-1} is bijective function

$\because$ X is a Hausdorff space

$\therefore$ $\exists$ open set U & V in X such that

$f^{-1}(x) \in U$, $f^{-1}(y) \in V$ & $U \cap V = \emptyset$

$\Rightarrow x \in f(U)$, $y \in f(V)$

$\because$ f is continuous and open

$\therefore$ $f(U)$, $f(V)$ are open sets in Y

Now $f(U) \cap f(V) = f(U \cap V)$

$= f(\emptyset)$

$= \emptyset$ $\because$ f is bijective.

Thus

$x \in f(U)$, $y \in f(V)$ & $f(U) \cap f(V) = \emptyset$

$\Rightarrow$ $f(X) = Y$ is Hausdorff.

So Hausdorffness is preserved under Homeomorphism i.e Hausdorffness is a top- property.

☆ The destindlion is always at a one step distance for a hardworker and patient. Ponder over it.

Theorem # A topological space X satisfying the first axiom of countability is a Hausdorff space iff every convergent sequence has a unique limit.

Proof # Let X be a Hausdorff space. Then every cgt sequence has necessarily unique limit.

Conversely let every cgt sequence has a unique limit and X is not Hausdorff. There must exist two points x and y such that every open set containing x does ~~not~~ intersect with every open set containg y. If $\{B_n(x)\}$ & $\{B_n(y)\}$ are monotone decreasing countable open bases at x and y respectively, we must have $B_n(x) \cap B_n(y) \neq \phi$ for every n. So we may choose a point x_n belonging to this intersection for each n. If G_x & G_y are arbitray open sets containg x & y respectively, then there must exist some integer N such that

$$B_n(x) \subseteq G_x \quad \text{and} \quad B_n(y) \subseteq G_y \quad \forall n > N$$

by the definition of monotone decreasing base. Hence $x_n \to x$ and $y_n \to y$ which is a contradiction. Hence X is a Hausdorff space.

Theorem # The topological product of two Hausdorff spaces is Hausdorff space.

Proof # Let (X, T_1) and (Y, T_2) be two Hausdorff spaces. We prove that $X \times Y$ is a Hausdorff space

Let (a,b), (c,d) be two different points in $X \times Y$

$\Rightarrow a, c \in X$ and $b, d \in Y$

$\because$ X and Y are Hausdorff spaces

$\therefore$ $\exists$ two open sets U_x, V_x in X and two open sets U_y, V_y such that

$a \in U_x$ $c \in V_x$ $U_x \cap V_x = \phi$ and $b \in U_y$, $d \in V_y$, $U_y \cap V_y = \phi$

Now $U_x \times U_y$ & $V_x \times V_y$ are open in $X \times Y$ and

$(a,b) \in U_x \times U_y \qquad (c,d) \in V_x \times V_y$

Also $(U_x \times U_y) \cap (V_x \times V_y) = \phi$

because if some point $(x,y) \in U_x \times U_y$

Then $x \in U_x$ & $y \in U_y$

$\Rightarrow x \notin V_x$ & $y \notin V_y$ $\quad (\because U_x \cap V_x = U_y \cap V_y = \phi)$

$\Rightarrow (x,y) \notin V_x \times V_y$

$\Rightarrow (U_x \times U_y) \cap (V_x \times V_y) = \phi$

$\Rightarrow X \times Y$ is Hausdorff space.

Theorem # $\prod_\alpha X_\alpha$ is Hausdorff iff each space X_α is Hausdorff.

Proof # Let (X_α, τ_α) be a non-empty Hausdorff space, for each $\alpha \in I$. Let $x = \{x_\alpha\}$, $y = \{y_\alpha\}$ be two distinct points in X. Since x and y are distinct, there must exist some index β such that $x_\beta \neq y_\beta$. Since X_β is a Hausdorff space, there must exist some open subsets G_x, G_y containing x_β and y_β respectively and $G_x \cap G_y = \phi$. Then $\pi_\beta^{-1}(G_x)$ and $\pi_\beta^{-1}(G_y)$ are disjoint open sets in the product space containing x and y respectively. Hence (X, T) is Hausdorff space

Conversely let $X = \prod_{\alpha \in I} X_\alpha$ be a Hausdorff space We shall show that each space X_α is Hausdorff by showing that X_α is homomorphic to a certain subset of of the product space and using hereditary property of Hausdorffness.

Fix a point $z = \{z_\alpha\} = \langle z_\alpha \rangle$ in (X, T), product space. Consider the subset E_β of the product space (X, T) consisting of all those points $\{x_\alpha\} = x$ such that $x_\alpha = z_\alpha$ if $\alpha \neq \beta$ while x_β may be any point of X_β

Let f be the restriction of projection mapping π_β restricted to E_β. Then f is clearly one-one, continuous mapping of E_β onto X_β since projection are continuous

Now a base for open sets in the subspace E_β is the family of intersections of E_β with the basis elements of the product topology. But $E_\beta \cap \prod_\alpha X_\alpha$

Comments # The construction used in the above theorem can be used to show that the set consisting of all points $\langle x_\alpha \rangle_\alpha$ such that $x_\alpha = z_\alpha$ if $\alpha \neq \beta_1, \beta_2, \ldots$ which we can denote by $E_{\beta_1} \times E_{\beta_2}$, is homeomorphic to $X_{\beta_1} \times X_{\beta_2}$. This argument may further be extended to any finite number of indices.

Theorem # The topological product $\prod_\alpha X_\alpha$ of T_0, T_1 spaces X_α is also T_0, T_1 respectively

Proof # Let x, y be two distinct points of $\prod_\alpha X_\alpha$. Then there is an index β such that

$$x_\beta = \bar{\pi}_\beta(x) \neq \bar{\pi}_\beta(y) = y_\beta$$

where $\bar{\pi}_\beta$ is the projection of $\prod_\alpha X_\alpha$ onto X_β. Hence x_β, y_β are two distinct points of X_β

Case-I Suppose that each X_α is a T_0-space. Then X_β is a T_0-space and there is an open U_β such that $x_\beta \in U_\beta$ and $y_\beta \notin U_\beta$

The set $G = \prod_\alpha G_\alpha$ where $G_\alpha = X_\alpha$ and $G_\beta = U_\beta$ $\alpha \neq \beta$ is such that $x \in G$ and $y \notin G$. Hence $\prod_\alpha X_\alpha$ is a T_0-space

Case-II # Suppose that each X_α is a T_1-space. Then for x_β, y_β in X_β, $x_\beta \neq y_\beta$ there exist open sets U_β and V_β such that

$x_\beta \in U_\beta$, $y_\beta \in V_\beta$ & $x_\beta \notin V_\beta$, $y_\beta \notin U_\beta$

So $G = \prod_\alpha G_\alpha$ $\quad H = \prod_\alpha H_\alpha$

when $G_\alpha = X_\alpha$, $\alpha \neq \beta$, $G_\beta = U_\beta$ and $H_\alpha = X_\alpha$, $\alpha \neq \beta$, $H_\beta = V_\beta$

are open sets in $\prod_\alpha X_\alpha$ such that

$x \in G$ $\quad y \notin G$ $\quad y \in H$, $x \notin H$

Hence $\prod X_\alpha$ is a T_1- space.

* Regular Space

A top- space X is regular if for any closed set F and point x of X not in F, there exist two disjoint open sets, one containing F and the other containing x i.e Any closed set F and a point not in F are strongly (dijo) separated.

Examples

(i) The indiscrete space is trivially a regular space. Here there is no closed F and a point x not in F because the only non-empty closed set is X itself. We know that indiscrete space is not T_0, T_1 and T_2. This example also shows that a regular space need not be T_0, T_1, T_2

(ii) Every discrete space is regular space because for any $a \in X$ & F closed set not containing a, $\exists$ two disjoint opensets $\{a\}$ & F such that

$a \in \{a\}$ $\quad$ & $F \subseteq F$

(∵ each subset of a discrete space is open as well as closed)

Theorem# (Regularity Criterion of Tychonov).
A topological space (X, T) is regular iff for every point $p \in X$, and every nbd U of p, there exists a nbd V of p, such that

$$\bar{V} \subseteq U .$$

Proof# Let (X, T) be a regular space, and U a nbd of a point $p \in X$. There exists an open set G such that

$$p \in G \subseteq U$$

Then $G^c = X - G$ is a closed set which does not contain p

$\because$ X is regular space

$\therefore$ there exist open set H and K such that $X - G \subseteq H$, $p \in K$ and $H \cap K = \phi$

Now from $H \cap K = \phi$, we have

$$K \subseteq X - H = H^c$$

$$\Rightarrow \bar{K} \subseteq \overline{X-H} = X - H \subseteq G \subseteq U \qquad \because X - G \subseteq H$$

$$G^c \subseteq H$$

$$\Rightarrow H^c \subseteq G$$

Then $k = V$ is an open nbd of the point p, such that $\bar{V} \subseteq U$

Conversely suppose that for every $p \in X$ and every nbd U of p, there exists a nbd V of p such that

$$\bar{V} \subseteq U$$

Let F be a closed set and p a point lying outside F. Then.

$U = X - F$ is an open nbd of p.

Hence by Hypothesis, there exists a nbd V of p, such that

$$\bar{V} \subseteq U = X - F$$

$\because$ V is a nbd of p

Therefore $\exists$ an open set G such that

$$p \in G \subseteq V$$

Then $G \cap (X-V) = \phi$

Also $V \subseteq \bar{V} \subseteq U = X-F$

$\Rightarrow V \subseteq F^c$

$\Rightarrow F \subseteq V^c = X-V$ and $p \in G$

Thus p and F are strongly seperated by G & $X-V$.

Hence (X,T) is a regular space

This Criterion may also be stated in the following equivalent forms.

Theorem #

(i) A topological space (X,T) is regular iff for every point $p \in X$, and every open set U containg p, there exists an open set V, such that

$$p \in V \subseteq \bar{V} \subseteq U$$

(ii) A topological space (X,T) is regular iff every nbd of p always encloses a closed nbd of p

Proof # (i) Let (X,T) be a regular space and p belongs to the open set U.

Then $F = U^c = X-U$ is closed set not containg p.

Since X is regular, $\exists$ open sets G_1, G_2 such that

$p \in G_1$, $U^c = F \subseteq G_2$ and $G_1 \cap G_2 = \phi$

Thus

$$p \in G_1 \subseteq G_2^c \subseteq U$$

$\because G_2^c$ is closed

$\therefore \bar{G_1} \subseteq \bar{G_2^c} = G_2^c$

Let $V = G_1$ Then

$\because G_1 \cap G_2 = \phi$

$\Rightarrow G_1 \subseteq G_2^c$

Also

$F = U^c \subseteq G_2$

$\Rightarrow G_2^c \subseteq U$

$$p \in V \subseteq \bar{V} \subseteq U$$

Conversely assume that $p \in U \in \tau \Rightarrow \exists\ V \in \tau$ such that

$$p \in V \subseteq \bar{V} \subseteq U$$

Let $p \in X$ and F be a closed set not containing p i.e $F \subseteq X - \{p\}$

Then $p \in X - F \in \tau$

Hence $\exists$ an open set V such that

$$p \in V \subseteq \bar{V} \subseteq X - F = F^c \qquad \bar{V} \subseteq F^c$$

$$\Rightarrow \quad F \subseteq (\bar{V})^c \in \tau \qquad \Rightarrow F \subseteq (\bar{V})^c_{\in \tau}$$

and $(\bar{V})^c \cap V = \phi \quad \because V \subseteq \bar{V}$

Thus

$$p \in V \in \tau \qquad F \subseteq \bar{V}^c \in \tau$$

and $V \cap (\bar{V})^c = \phi$

Hence (X, τ) is regular space.

(ii) Let (X, τ) be a regular space and U a nbd of point $p \in X$. There exists an open set G such that

$$p \in G \subseteq U$$

Then $G^c = X - G$ is a closed set not containing p

$\because$ X is regular

$\therefore$ $\exists$ an open sets H & K such that

$X - G \subseteq H \qquad p \in K$ and $H \cap K = \phi$

Now from $H \cap K = \phi$

we have $K \subseteq H^c$ where H^c is closed

$\Rightarrow \quad p \in K \subseteq H^c$

$\Rightarrow \quad H^c$ is a closed nbd of p

Also from $X - G \subseteq H$, we have

$$H^c \subseteq G$$

$$\Rightarrow \quad p \in H^c \subseteq G \subseteq U$$

$\Rightarrow$ U contains a closed nbd H^c of p

Convesely suppose that for any point p of X, every nbd U of p encloses a closed nbd of p.

Let F be a closed set not containg p

Then $U = X - F$ is an open nbd of p.

Hence by Hypothesis U contains a closed nbd K of p i.e

$$K \subseteq U$$

$\because$ K is a nbd of p

$\therefore$ $\exists$ an open set H containg p such that

$$p \in H \subseteq K$$

$$\Rightarrow H \subseteq K \subseteq U$$

Now $F \not\ni U \Rightarrow F \subseteq U^c \subseteq K^c \in \tau$

$\Rightarrow p \in H$ $\quad F \subseteq K^c \in \tau$ & $H \cap K^c = \phi$

$\Rightarrow (X, \tau)$ is a regular space.

Theorem # A top-space (X, T) is regular iff any one of the following conditions holds

(i) # For any closed set F and a point $p \notin F$, there exists a nbd V of p such that $F \cap \bar{V} = \phi$

(ii) # If A and B are pair of subsets of X satisfying $A \cap B \neq \phi$, Then then there exists an open set G Such that $A \cap G \neq \phi$ and $\bar{G} \subseteq B$

(iii) For every non-empty subset A and a closed subset B satisfying $A \cap B \neq \phi$, there exists a pair of disjoint open sets U, V such that

$$A \cap U \neq \phi \text{ and } B \subseteq V$$

Proof # (i) Let (X, τ) be a regular space. and F be any closed set not containg a pot p.

Then $\exists$ open set G and H such that

$p \in G$ $\quad F \subseteq H$ & $G \cap H = \phi$

$\Rightarrow G \subseteq H^c$, where H^c is closed

$\Rightarrow p \in G \subseteq H^c$

$\Rightarrow H^c = V$ is a nbd of p Also $\overline{H^c} = \overline{V} = V$

Now since $F \subseteq H$, Therefore $F \cap H^c = \phi$

$\Rightarrow F \cap V = \phi \Rightarrow F \cap \overline{V} = \phi$

Condition is proved.

Conversely let for any closet F and a point $p \notin F$ there exists a nbd V of p such that

$$F \cap \overline{V} = \phi$$

Then since $V \subseteq \overline{V}$

Therefore $F \cap V = \phi$

from $F \cap \overline{V} = \phi$, we have

$F \subseteq (\overline{V})^c = G \in \tau$ & $(\overline{V})^c \cap V = \phi$

$\Rightarrow p \in V$ $F \subseteq G$ & $V \cap G = \phi$

$\Rightarrow (X, T)$ is a regular space.

(ii) # Let (X, T) be a regular space and suppose that for a pair of set A, B satisfying $A \cap B \neq \phi$, there is no open set G which is such that

$$A \cap G \neq \phi \quad , \quad \overline{G} \subseteq B$$

But every open set is such that which violates these Condition.

Let $p \in A \cap B$ and F be any closed set not containg P.

Then $p \in A$, $p \in B$

$\because X$ is regular & $p \in X - F$ an open nbd of p

$\therefore \exists$ a nbd V of p such that

$$p \in V \subseteq \overline{V} \subseteq X - F = F^c$$

Now ~~$B \subseteq \overline{V}$~~ (as per ~~supposition~~

Now F^c is an open set containg p and so intersects A. which is a contradiction. Therefore Condition is true

Conversely suppose that condition is true.

Theorem# A topological space $(X, \mathcal{T})$ is regular iff any one of the following conditions holds

(i) # The closed nbds of any point $p \in X$ form a base about the point p

(ii) For every closed subset A, the intersection of all closed nbds of A in $(X, \mathcal{T})$ is the set A

[V is called a nbd of a set A, if $A \subseteq Int(V)$]

Proof# Let $(X, \mathcal{T})$ be a regular space and G an open set, containing the point p. Then $X - G$ is a closed set and $p \notin X - G$.

$\because$ X is regular

$\therefore$ $\exists$ two open sets U & V such that

$p \in U$ $\quad X - G \subseteq V$ and $U \cap V = \phi$

$\Rightarrow p \in U \subseteq X - V \subseteq G$

Thus G contains a closed nbd of p. | $G' \subseteq V$ $\Rightarrow V' \subseteq G$

Hence the closed nbds of p form a local base at the point p

Conversely let the closed nbds a point $p \in X$ form a local base at the point p, for every p in $(X, \mathcal{T})$ let F be a closed set and p any point lying out side of F. Then $X - F$ is an open set, containing p. Hence by the hypothesis $\exists$ a closed nbd N of p s.t

$$N \subseteq X - F.$$

As N is a closed nbd of p, there exists an open set G such that

$$p \in G \subseteq N$$

Then $G \cap X - N = \phi$ | $N \subseteq F^c$ $\Rightarrow F \subseteq N^c$

$\Rightarrow p$ & F are separated two disjoint open sets G & $X - N$

Hence $(X, \mathcal{T})$ is a regular space

(ii) Let $(X.T)$ be a regular space and A a closed subset in $(X.T)$. Let p be any point lying in A^c. Then $A^c = X - A$ is an open nbd of p. Hence by regularity criterion, there exists an open set G such that

$$p \in G \subseteq \bar{G} \subseteq X - A = A^c$$

$$\Rightarrow A \subseteq X - \bar{G} \subseteq X - G$$

$\Rightarrow X - G = G^c$ is a closed nbd of the set A and such that $p \notin X - G$

$\Rightarrow$ Every closed nbd contains only A but A^c

Hence intersection of all closed nbds of A is the set A

Conversely let the condition be satisfied in $(X.T)$. Let A be a closed and p be a point not in A. Then since intersection of all closed nbds of A is A and $p \notin A$, therefore $\exists$ a closed nbd V of A such that $p \notin V$

$\because$ V is a nbd of A

$\therefore$ $A \subseteq \text{Int}(V) \subseteq V$ & $p \in X - V$

$\Rightarrow$ A and p are strongly separated by two openst $\text{Int}(V)$ and $X - V$. Hence $(X.T)$ is a regular space

Theorem # For any two distinct points p, q in a regular space either $\overline{\{p\}} = \overline{\{q\}}$ or $\overline{\{p\}} \cap \overline{\{q\}} = \phi$ i.e closures of two distinct singletons are either identical or distinct.

Proof # Let p, q be any two distinct points in a regular space $(X.T)$.

If $p \in \overline{\{q\}}$ and $q \in \overline{\{p\}}$, Then

$$\overline{\{p\}} \subseteq \overline{\overline{\{q\}}} = \overline{\{q\}} \subseteq \overline{\overline{\{p\}}} = \overline{\{p\}}$$

$$\Rightarrow \overline{\{p\}} = \overline{\{q\}}$$

Let one of the relations $p \in \overline{\{q\}}$ and $q \notin \overline{\{p\}}$ does not hold say $q \notin \overline{\{p\}}$.

Then the closed set $\overline{\{p\}}$ and the point q not in $\overline{\{p\}}$ must be (regular) strongly separated in (X, T).

Hence $\exists$ an open set G such that

$$\overline{\{p\}} \subseteq G \quad \text{and} \quad q \in X - G \qquad \because q \notin \overline{\{p\}} \subseteq G$$

So that

$$\overline{\{q\}} \subseteq \overline{X - G} = X - G \qquad \because X - G \text{ is closed}$$

$$\Rightarrow \overline{\{p\}} \cap \overline{\{q\}} = \phi$$

Thus either $\overline{\{p\}} = \overline{\{q\}}$ or $\overline{\{p\}} \cap \overline{\{q\}} = \phi$

Theorem # The following statements about a topological space X are equivalent

(a) # X is a regular space

(b) # For any open set U in X and $x \in U$, there is an open set V containing x such that

$$x \in \overline{V} \subseteq U$$

(c) # Each element of X has a (local) neighbourhood base consisting of closed sets.

Proof # (a) $\Rightarrow$ (b)

Suppose that X is regular. Let U be an open set and $x \in U$. Then $U^c = X \setminus U$ is a closed set not containing x.

$\because X$ is regular

$\therefore \exists$ open set V & V_1 such that

$x \in V$ $\quad X \setminus U \subseteq V_1$ & $V \cap V_1 = \phi$

$\Rightarrow X - V_1 \subseteq U$ & $V \subseteq X \setminus V_1 \subseteq U$

so that

$$\overline{V} \subseteq \overline{X \setminus V_1} = X - V_1 \subseteq U \quad \text{i.e}$$

$x \in V$ and $\overline{V} \subseteq U$

(b) $\Rightarrow$ (c)

Suppose that (b) is true.

Let y be any element in X. Then for any open set U containing y, there is an open set V such that

$$y \in V \quad , \quad \bar{V} \subseteq U$$

$\because y \in V \subseteq \bar{V}$

Thus for each $y \in X$, the sets of the form $\bar{V}$ form a closed nbd base consisting of closed subsets of X. i.e (C) is satisfied.

(C) $\Rightarrow$ (a) Suppose that every point x in X has a closed nbd base. Let A be a closed set not containing x. Then $x \in X \setminus A$ i.e $X \setminus A$ is an open nbd of x. By (C), There is a closed set B which contains x and is a closed nbd of x such that

$$B \subseteq X \setminus A = A^c \quad \Rightarrow A \subseteq X \setminus B$$

Then $x \in B$, $A \subseteq X - B = V$

and $B \cap V = \phi$

Therefore X is regular space

Theorem# Every subspace of a regular space X is a regular space i.e the property of being a regular space" is a hereditary property.

Proof# Let Y be a subset of regular space X and F be a closed set in Y and y be a point in (Y, τ_Y) which is not in F.

Since closure of F in X ($\bar{F}_x$) is the smallest closed super set of of F (i.e intersection of all closed supersets of F), therefor $y \notin \bar{F}_x$

$\Rightarrow \bar{F}_x$ is a closed set not containg $y \in X \quad \because Y \subseteq X$

$\because X$ is regular

$\therefore \exists$ two disjoint open subsets in X, G, H such that $y \in G$ and $\bar{F}_x \subseteq H$

$\because G$ & H are open in X

$\therefore$ $Y \cap G$ & $Y \cap H$ are open in Y
and $y \in Y \cap G$, $F \subseteq F_x \subseteq H$

But $F \subseteq Y$
$\Rightarrow F \subseteq Y \cap H$

Moreover

$$(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) \quad \text{associative Law}$$
$$= Y \cap \emptyset$$
$$= \emptyset$$

Hence (Y, T_Y) is regular

T_3 – Space

A top-space which is both regular and T_1- space is called a T_3– Space
A Regular T_1 space is called a T_3-space

Warning # Some auther use exactly the opposite Convention w.r.t words "regular" and "T_3" i.e they call regular space the space we have said T_3-space and our regular space as T_3– space. While some author do not use either term unless T_1– axiom is satisfied.

Although every T_3 space – T_2–space, a regular space need not be a T_2–space and T_2–space need not to be a regular (T_3) space
This can be seen from the example

Example # Let R denote the set of real numbers and Q the subset, formed by the rational numbers. Let S be the collection of all open intervals and the subset Q: then S is a sub-base of topology T_1 on R. Since the usual topology T on R is a Hausdorff topology, and T_1 is finer than T, it follows that (R, T_1) is a Hausdorff space. But

(R, T_1) is not a regular space, since the closed set $R-Q$ (i.e the set of irrational numbers) and the point $\frac{1}{2}$, lying outside, lying out side $R-Q$, can not be separated strongly. Thus

(i) (R, T_1) is a Hausdorff space, but not regular space

(ii) the usual topology T on R is regular, but the topology T_1 on R which is finer than T is not regular.

Any indiscrete space with more than two 2 elements is a regular space, which is not T_3 - space

Theorem # Any T_3 - space is a T_2 - space.

Proof # Let (X, T) be a T_3 - space. Then (X, T) is both regular and T_1 - space. Let $x \neq y$ be any two distinct points in (X, T). Since (X, T) T_1-space, one paintic set in T_1 - space are closed, $\{x\}$ is closed in (X, T). Also $y \notin \{x\}$. Since (X, T) is regular, y, $\{x\}$ can be strongly separated by open sets. Thus the points x & y can be strongly separated in (X, T) and (X, T) is therefore a T_2-space

Theorem # Every Regular T_0-space is T_1 and is therefore T_3 - space

Proof # Let (X, T) be a regular space and p be any point in X. Let q be any point in X different from p.

$\because$ (X, T) is a T_0 - space

$\therefore$ either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$

$\Rightarrow$ $p \neq q \Rightarrow \overline{\{p\}} \neq \overline{\{q\}}$

$\because$ (X, T) is regular space

$\therefore$ either $\overline{\{p\}} = \overline{\{q\}}$ or $\overline{\{p\}} \cap \overline{\{q\}} = \emptyset$

But we have above that $\overline{\{p\}} \neq \overline{\{q\}}$, Therefore we have

$\{\bar{p}\} \cap \{\bar{q}\} = \phi$

$\Rightarrow q \notin \{\bar{p}\}$

which means $\{\bar{p}\}$ doesnot contain any point q, different from p i.e $\{\bar{p}\} = \{p\} \Rightarrow \{p\}$ is closed.

Thus every one pointic set in $(X, \mathcal{T})$ is closed so that $(X, \mathcal{T})$ is necessarily a T_1-space. And as it is also regular, it is a T_3-space.

Theorem # The following properties hold in T_3-space $(X, \mathcal{T})$

(i) Each pair of distinct points in X have nbds, whose closures are disjoint

(ii) If A is an infinite subset of X, then there exists an infinite sequence $\{V_i : i = 0, 1, \ldots\}$ of open subsets of X, such that $\bar{V}_m \cap \bar{V}_n = \phi$ for $m \neq n$, and $A \cap V_n \neq \phi$ for each $n \geq 1$

Proof # (i) Let p, q be any two distinct points in X. Since $(X, \mathcal{T})$ is a T_3-space and therefore also a T_1-space, singleton are closed sets and $\{q\}$ is a closed set. Now $\{q\}$ is a closed set not containing p. So $p \in X - \{q\} \in \mathcal{T}$

$\because$ $(X, \mathcal{T})$ is regular

$\therefore$ There exists an open set U such that

$$p \in U \subseteq \bar{U} \subseteq X - \{q\}$$

$\Rightarrow q \in X - \bar{U}$

$\because \bar{U} \subseteq \{q\}^c$

$\Rightarrow \{q\} \subseteq X - \bar{U}$

and there exists an open set V, such that

$$q \in V \subseteq \bar{V} \subseteq X - \bar{U}$$

Now U & V are open nbds of p & q such that their closures $\bar{U}$ & $\bar{V}$ are disjoint

(ii) We prove this by induction. Take $V_0 = \phi$ and assume that the open sets $V_0, V_1, \ldots V_m$ have been obtained such that $\bar{V}_0, \bar{V}_1, \ldots \bar{V}_m$

are pairwise disjoint, $A \cap V_k \neq \phi$ for $k = 1, 2 \ldots m$ and

$$A_m = A - [\bar{V}_0 \cap \bar{V}_1 \cap \ldots \cap \bar{V}_m]$$

is an infinite set. Let p, q be two distinct points in A_m. By regularity Criterion, there exists an open set V such that

$$p \in V \subseteq \bar{V} \subseteq X - [\bar{V}_0 \cup \bar{V}_1 \cup \ldots \cup \bar{V}_m \cup \{q\}]$$

and also an open set W such that

$$q \in W \subseteq \bar{W} \subseteq X - [\bar{V}_0 \cup \bar{V}_1 \cup \ldots \cup \bar{V}_m \cup \bar{V}]$$

Let $V_{m+1} = V$ if $A \cap V$ is finite and $V_{m+1} = W$ if $A \cap V$ is infinite

Then V_{m+1} is an open set such that $\bar{V}_0, \bar{V}_1 \ldots \bar{V}_m, \bar{V}_{m+1}$ are pairwise disjoint and $A \cap V_k \neq \phi$ for $k = 1, 2 \ldots m+1$ Also the set

$$A - [\bar{V}_0 \cup \bar{V}_1 \cup \ldots \cup \bar{V}_{m+1}] \text{ is infinite}$$

This completes the proof by principle of mathematical induction.

Note The converse of (i) is not true in general

Normal Space

A top-space X is normal if each pair of two disjoint closed sets can be strongly separated i.e if for any two disjoint closed sets $F_1 \& F_2$ there exist two open sets $U \& V$ such that

$$F_1 \subseteq U \quad F_2 \subseteq V \qquad U \cap V = \phi$$

or equivalently two distinct closed sets are separated by open sets in the sense that their nbds which are disjoint

Warning # The definition of normal space is not standard. Some topologists consider T_1- property essential for normal space while other do not consider it essential for normal space.

There are several necessary and sufficient conditions for a topological space to be normal e.g Urysohn's Lemma and Tietze's theorem.

Theorem (Normality criterion of Urysohn):
A topological space $(X, \mathcal{T})$ is normal iff for every closed set F and any open set U containing F, $\exists$ an open set V, such that

$$F \subseteq V, \quad \bar{V} \subseteq U$$

Proof# Let $(X, \mathcal{T})$ be a normal space, and U any open set containing a closed set F. Then F and $U^c = X - U$ are disjoint closed sets.
$\because$ $(X, \mathcal{T})$ is a normal space
$\therefore$ $\exists$ Two open sets V and G such that
$F \subseteq V$ $\quad X - U \subseteq G$ and $V \cap G = \phi$
Now $V \cap G = \phi$ implies that

$$V \subseteq X - G$$

$$\Rightarrow \bar{V} \subseteq \overline{X - G} = X - G \subseteq U \qquad \because X - G \text{ is closed}$$

Thus $F \subseteq V$, $\bar{V} \subseteq U$
Conversely let the given condition be true. Let F_1 and F_2 be disjoint closed sets in $(X, \mathcal{T})$.
Then $F_1 \cap F_2 = \phi$

$$\Rightarrow F_1 \subseteq X - F_2 = F_2^c \in \mathcal{T}$$

i.e F_2^c is an open set containing a closed set F_1.
Hence by condition $\exists$ an open set V such that

$$F_1 \subseteq V \text{ and } \bar{V} \subseteq X - F_2$$

$$\Rightarrow F_1 \subseteq V \text{ and } F_2 \subseteq (\bar{V})^c = X - \bar{V}$$

Now $V \cap (X - \bar{V}) = \phi$
Thus F_1 & F_2 are strongly separated by disjoint open sets V & $X - \bar{V}$. Hence $(X, \mathcal{T})$ is normal space.

Corollary # A top-space (X, T) is normal iff one of the following conditions holds.

1) For each pair of disjoint closed sets P, Q, $\exists$ an open set U, such that $P \subseteq U$ and $Q \cap \bar{U} = \phi$

2) Each pair of disjoint closed sets have nbds, whose closures are disjoint.

Proof # (1) Let $(X\ T)$ be normal and P, Q. be any two disjoint closed sets. Then $\exists$ two open sets G, H such that

$$P \subseteq G \qquad Q \subseteq H \cdot \text{ and } \quad G \cap H = \phi$$

Now G is an open set containing a closed set P and by above Theorem $\exists$ an open set U such that

$$P \subseteq U \subseteq \bar{U} \subseteq G$$

$$\text{Now } \bar{U} \subseteq G \qquad Q \subseteq H \quad \& \; G \cap H = \phi$$

$$\Rightarrow Q \cap \bar{U} = \phi$$

Thus

$$P \subseteq U \qquad \text{and} \quad Q \cap \bar{U} = \phi$$

Conversely suppose that the condition is true. Let F_1 and F_2 be any two disjoint closed sets. Then $\exists$ an open set U such that

$$F_1 \subseteq U \qquad \& \; F_2 \cap \bar{U} = \phi$$

Now F_2 & $\bar{U}$. are two disjoint closed sets and by Condition $\exists$ and open set V such that

$$F_2 \subseteq V \qquad \& \quad \bar{U} \cap \bar{V} = \phi$$

$$\Rightarrow \overline{U \cap V} \subseteq \bar{U} \cap \bar{V} = \phi$$

$$\text{But } \phi \subseteq \overline{U \cap V}$$

$$\Rightarrow \overline{U \cap V} = \phi$$

$$\Rightarrow U \cap V = \phi \qquad \because \bar{\phi} = \phi$$

Thus F_1 & F_2 are strongly separated and so (X, T) is normal

2) Let (X, T) be a normal space and F_1, F_2 be a pair of disjoint closed sets. There exist open sets G & H such that

$F_1 \subseteq G$ $\quad$ $F_2 \subseteq H$ $\quad$ & $G \cap H = \phi$

By above Theorem $\exists$ open sets U & V such that

$F_1 \subseteq U \subseteq \bar{U} \subseteq G$ $\quad$ $F_2 \subseteq V \subseteq \bar{V} \subseteq H$

$\because$ $G \cap H = \phi$

$\therefore$ $\bar{U} \cap \bar{V} = \phi$

Hence U & V are nbds of F_1 & F_2, whose closures are disjoint

Conversely suppose that for any pair of disjoint closed sets $\exists$ nbds whose closures are disjoint

Let F_1 & F_2 be two disjoint and N_1, N_2 their nbds such that $\bar{N_1} \cap \bar{N_2} = \phi$

$\because$ N_1 is a nbd of F_1

$\therefore$ $\exists$ an open set U s.t

$F_1 \subseteq U \subseteq N_1$

Similarly $\exists$ an open set V s.that

$F_2 \subseteq V \subseteq N_2$

Now $\bar{U} \subseteq \bar{N_1}$ & $\bar{V} \subseteq \bar{N_2}$

$\because$ $\bar{N_1} \cap \bar{N_2} = \phi$

$\Rightarrow$ $\bar{U} \cap \bar{V} = \phi$

Now $F_1 \subseteq U \subseteq \bar{U}$ $\quad$ $F_2 \subseteq V \subseteq \bar{V}$

$\Rightarrow$ $U \cap V = \phi$

$\Rightarrow$ F_1 & F_2 are strongly separated.

Hence (X, T) is a normal space.

Examples

(1) Every discrete space with at least two points is normal

Scroll to next page

(2) Every Metric Space is normal

Usually subspace of any normal space is not normal but a closed space is always a normal.

Theorem

Proof Let A & B be disjoint closed subsets of metric space (X, d). If A or B is trivial, then ϕ and X are the required open set containg A and B and ϕ, X are disjoint. Suppose that A & B are non-empty. Let $a \in A$. Then $a \notin B$. Since B is closed

$$d(a, B) = \inf_{b \in B} d(a, b) > 0$$

Let $r_a = d(a, B)$. Similarly for every point b of B, $b \notin A$ and

$$r_b = d(b, A) > 0$$

Consider now the open balls

$$B(a, \tfrac{r_a}{3}) \; ; \quad B(b, \tfrac{r_b}{3})$$

Let $U = \bigcup_{a \in A} B(a, r_a/3)$, $V = \bigcup_{b \in B} B(b, r_b/3)$

Then U & V being union of open sets are open sets. Also

$$A \subseteq U \qquad B \subseteq V$$

Let $x \in U \cap V$

Then there are points a_1 and b_1 in A & B respectively such that

$$x \in B(a_1, r_1/3) \qquad x \in B(b_1, r_1'/3)$$

$$\Rightarrow d(a_1, x) < r_1/3 \qquad d(b_1, x) < r_1'/3$$

Since A & B are closed and disjoint,

$d(a_1, b_1) = r > 0$ with $d(a_1, B) = r_1 \leq r$ $\quad d(b_1, A) = r_1' \leq r$

By definition of infimum

$$r = d(a_1, b_1) \leq d(a_1, x) + d(x, b_1)$$

$$< r_1/3 + r_1'/3 < \frac{2r}{3} < r$$

which is a contradiction. Hence $U \cap V = \phi$

Hence (X, d) is normal.

Theorem# A closed subspace of a normal space is normal.

Proof# Let A be a closed subspace of a normal space X. and A_1, A_2 be disjoint closed subsets of A. Then there are closed sets B_1, B_2 in X such that

$A_1 = A \cap B_1$ $\qquad A_2 = A \cap B_2$

$\because$ A is closed in X

$\therefore$ A_1, A_2 are also closed in X

$\because$ X is a normal space

$\therefore$ $\exists$ open sets U_1 & U_2 such that

$A_1 \subseteq U_1$ $\qquad A_2 \subseteq U_2$ $\qquad U_1 \cap U_2 = \phi$

$\Rightarrow A_1 \cap A \subseteq U_1 \cap A$ $\qquad A_2 \cap A \subseteq U_2 \cap A$

$\Rightarrow A_1 \subseteq A \cap U_1$ $\qquad A_2 \subseteq A \cap U_2$ $\qquad \because A_1 \subseteq A, A_2 \subseteq A$

Now $(A \cap U_1) \cap (A \cap U_2)$

$= A \cap (U_1 \cap U_2)$ associative laws

$= A \cap \phi = \phi$

$\Rightarrow A \cap U_1$ & $A \cap U_2$ are open and disjoint in A.

$\Rightarrow$ A is normal

Theorem# Every regular Lindelof space is normal

Proof# Let F_1 & F_2 be two disjoint closed subsets of the regular Lindelof space X.

$\because$ Every closed subset of Lindelof space is Lindelof

$\therefore$ F_1 & F_2 are Lindelof

Now $F_1 \cap F_2 = \phi \Rightarrow F_1 \subseteq X \setminus F_2 = F_2^c$

where $X \setminus F_2$ is an open set

Since X is a regular space, for each point $x \in F_1 \subseteq F_2^c$ there exists an open set $G(x)$ such that

$x \in G(x) \subseteq \overline{G(x)} \subseteq X \setminus F_2$

$\Rightarrow \bigcup_{x \in F_1} \{x\} \subseteq \bigcup \{G(x) : x \in F_1\}$

$F_1 \subseteq \bigcup \{G(x) : x \in F_1\}$

$\Rightarrow$ The collection $\{G(x) : x \in F_1\}$ forms an open covering of the Lindelof set F_1.

$\because$ F_1 is Lindelof set

$\therefore \exists$ a countable sub-cover $\{G_i\}_{i \in N}$.

Similarly for each $x \in F_2 \subseteq X - F_1$, there exists an set $G^*(x)$ such that

$$x \in G^*(x) \subseteq \overline{G^*(x)} \subseteq X - F_1$$

The collection $\{G^*(x) : x \in F_2\}$ forms an open covering of the Lindelof space F_2

There must be a countable sub-covering $\{G_i^*\}_{i \in N}$

Now $G_n \setminus \bigcup \overline{G}_{i \le n} \subseteq F_2^c$ & $G_n^* \setminus \bigcup \overline{G}_{i \le n} \subseteq F_1^c$ $\forall n$

$\Rightarrow G = \bigcup (G_n \setminus \bigcup \overline{G}_{i \le n}^*) \subseteq F_2^c$ & $G^* = \bigcup \left[G_n^* \setminus \bigcup \overline{G}_{i \le n} \right] \subseteq F_1^c$

$\Rightarrow G \cap G^* = \phi$

Also

$F_1 \subseteq G$ & $F_2 \subseteq G^*$

Thus the space X is normal

$f : X \to Y$ If closed the image is also closed.

Closed cont: f closed $\Rightarrow$ image

Theorem # A closed continuous image of a normal space is normal

Proof # Let X be a ~~normal~~ space and $Y = f(X)$ is a continuous image of X, where f is closed i.e f maps closed sets into closed sets

Set A_1, B_1 be closed disjoint subsets of $Y = f(X)$.

Then by continuity of f

$A = f^{-1}(A_1)$ $\quad B = f^{-1}(B_1)$

are closed subsets of X and also there are disjoint.

$A \cap B = f^{-1}(A_1) \cap f^{-1}(B_1)$

$= f^{-1}(A_1 \cap B_1) = f^{-1}(\phi) = \phi$.

disjoint.

under continuity inverse image of open is closed & vice versa

$\because$ X is normal

$\therefore \exists$ disjoint open sets U & V such that

$A \subseteq U$ $\quad B \subseteq V \Rightarrow A_1 \subseteq f(U)$, $B_1 \subseteq f(V)$

Now $U^c = X \setminus U$ closed, $V^c = X \setminus V$ closed

are closed in X

$\Rightarrow f(U^c)$, $f(V^c)$ are closed in $Y = f(X)$

Thus $U_1 = Y - f(U^c)$, $V_1 = Y - f(V^c)$ are open, disjoint and contain B_1 & A_1 respectively

Hence $Y = f(X)$ is normal

$U \cap V = \varphi$ & $U \cup V = X$

$U^c \cup V^c = X$.

T_4: Space # A normal T_1-space is called T_4-space.

Theorem # A T_4-space is regular space.

Proof # Let X be a T_4-space. Then X is T_1-space and normal.

Let F be a closed set and x a point not in F.

$\because$ X is T_1-space

$\therefore$ Singletor $\{x\}$ is closed set

Also $F \cap \{x\} = \varphi$ $\quad \because x \notin F$

$\because$ X is normal

$\therefore$ F & $\{x\}$ are strongly separated.

$\Rightarrow \exists$ open sets U & V such that

$F \subseteq U$ & $\{x\} \subseteq V$ i.e $x \in V$

and $U \cap V = \varphi$

$\Rightarrow X$ is regular space

Comments # Every T_4-space is T_2-space but normal space need not regular. A normal space is not a Hausdorff, of Course, although it is a T_0-space.

Coverings

A Collection $C = \{A_\alpha : \alpha \in I\}$ of subsets of a set X such that $\cup\{A_\alpha : \alpha \in I\} = X$ is called a covering (or cover) of the set X. If A subcollection C_1 of C is also a covering of X, then C_1 is called a sub-covering of the covering C.

Open Covering

Let (X.T) be a topological space, then a covering C (respectively, a sub-covering C_1) is called an open covering (respectively an open sub-covering) of X if the members of C are open sets in (X.T). like-wise we define a closed covering and closed sub covering.

Covering of a Subset of Top-space #.

If E is a subset of X, then a collection $\{A_\alpha : \alpha \in I\}$ of subsets of X is called a covering of E if

$$E \subseteq \cup\{A_\alpha : \alpha \in I\}$$

Shrinkable Open Covering

Let $C = \{G_\alpha : \alpha \in I\}$ be an open covering of X. Let $H = \{H_\alpha : \alpha \in I\}$ be also an open covering of X such that

$$\overline{H_\alpha} \subseteq G_\alpha \quad \forall \alpha \in I$$

Then the open covering C is said to be shrinkable and the open covering H is called a shrinking of the open covering C.

Point-Finite Covering

A covering C of a top-space (X.T) is said to be point-finite if each $x \in X$ belongs

only finitely many members of C

Theorem# A top-space (X, T) is normal iff every finite open covering of X is shrinkable

Proof# Let (X, T) be a normal space and $C = \{G_1, G_2, \dots, G_n\}$ be a finite covering of X.

Then

$$X = \bigcup_{i=1}^{n} G_i$$

$$\Rightarrow X^c = \left(\bigcup_{i=1}^{n} G_i\right)^c$$

$$\phi = \bigcap_{i=1}^{n} G_i^c$$

$$= G_1^c \cap G_2^c \cap G_3^c \cap \dots \cap G_n^c$$

Now G_1^c and $(G_2^c \cap G_3^c \cap \dots \cap G_n^c)$ form a pair of disjoint closed sets in (X, T)

$\because (X, T)$ is a normal space

$\therefore$ ∃ two open sets V_1 & H_1 such that $G_1^c \subseteq V_1$, $\bigcap_{i=2}^{n} G_i^c \subseteq H_1$ and $V_1 \cap H_1 = \phi$

$$\Rightarrow H_1 \subseteq X - V_1 = V_1^c$$

$$\Rightarrow \overline{H_1} \subseteq \overline{X - V_1} = X - V_1 \subseteq G_1 \qquad \because G_1^c \subseteq V_1$$

$$\Rightarrow \overline{H_1} \subseteq G_1 \qquad \therefore V_1^c \subseteq G_1$$

Again Consider

$$\bigcap_{i=2}^{n} G_i^c = G_2^c \cap G_3^c \cap G_4^c \cap \dots \cap G_n^c$$

$$\Rightarrow \because \bigcap_{i=2}^{n} G_i^c \subseteq H_1$$

$$\Rightarrow (X - G_2) \cap \dots (X - G_n) \subseteq H_1$$

$$\Rightarrow \bigcap_{i=2}^{n} G_i^c \cap X - H_1 = \phi$$

~~Complementing on both sides~~

$$\Rightarrow \left(\bigcap_{i=2}^{n} G_i^c\right) \cap H_1^c = \phi$$

$\Rightarrow \left(\bigcup_{i=2}^{n} G_i\right)^c \cap H_1^c = \phi$

Complementing on both sides

$$\left[\left(\bigcup_{i=2}^{n} G_i\right)^c \cap H_1^c\right]^c = \phi^c$$

$$\bigcup_{i=2}^{n} G_i \cup H_1 = X$$

$\Rightarrow \{H_1, G_2, G_3 - - - - , G_n\}$ forms an open covering of X where $\bar{H}_1 \subseteq G_1$.

By continuing the above process, we can replace $G_2, G_3 - - - G_n$, successively by open sets $H_2, H_3 - - - H_n$ and thus obtain a shrinking $H = \{H_1, H_2, - - - , H_n\}$ of the given open covering C

Conversely let every finite open covering of X shrinkable, for a top-space (X, T).

Let P, Q be a pair of disjoint closed sets in (X, T). Then

$$P \cap Q = \phi$$

$$\Rightarrow P^c \cup Q^c = X$$

$\Rightarrow \{P^c, Q^c\}$ is a finite open covering of X. Hence by Hypothesis, there exists a shrinking $\{G, H\}$ of the covering $\{P^c, Q^c\}$ i.e

$G \cup H = X$ $\quad \bar{G} \subseteq P^c$ & $\bar{H} \subseteq Q^c$

Now $\bar{G} \subseteq P^c$ $\quad \bar{H} \subseteq Q^c$

$\Rightarrow P \subseteq X - \bar{G}$ $\quad Q \subseteq X - \bar{H}$

and $(X - \bar{G}) \cap (X - \bar{H}) = X - (\bar{G} \cup \bar{H})$

$= X - X = \phi$

~~$\{\bar{G}, \bar{H}\}$ is a covering~~

$\{\because \bar{G} \supseteq G \quad \bar{H} \supseteq H$

$\Rightarrow \bar{G} \cup \bar{H} \supseteq G \cup H = X\}$

Thus P and Q are separated strongly by the disjoint open sets $X - \bar{G}$ and $X - \bar{H}$. Hence (X, T) is normal

Comments # The theorem can be generalised for a particular type of infinite open coverings as is proved in the theorem below

Theorem # A topological space $(X, \mathcal{T})$ is normal iff every point-finite open covering of X is shrinkable

Proof # Let $(X, \mathcal{T})$ be a normal space and let $C = \{G_\alpha : \alpha \in I\}$ be a point-finite covering of X. Let the indexing set I be well ordered and for convenience let $I = \{1, 2, \ldots \alpha, \ldots\}$. We now construct a collection of subsets $H = \{H_\alpha : \alpha \in I\}$ by transfinite induction as follows

Let $F_1 = X - \cup \{G_\lambda : \lambda > 1\}$: then F_1 is a closed set, contained in the open set G_1.

By normality criterion, there exists an open set H_1 such that

$$F_1 \subseteq H_1 \quad \text{and} \quad \overline{H_1} \subseteq G_1$$

Let the sets H_μ be defined as above, for each $\mu < \alpha$.

Let now $F_\alpha = X - [\cup \{H_\mu : \mu < \alpha\}] \cup [\cup \{G_\nu : \nu > \alpha\}]$

Then F_α is a closed set contained in G_α.

Hence by normality criterion, there exists an open set H_α such that

$$F_\alpha \subseteq H_\alpha \qquad \overline{H_\alpha} \subseteq G_\alpha$$

Then the family $H = \{H_\alpha : \alpha \in I\}$ forms a shrinking of the given (point-finite covering C, provided H forms a covering of X. Let $x \in X$. Then x belongs to only finitely many members of C say $G_{\alpha_1}, G_{\alpha_2} \ldots, G_{\alpha_n}$

Let $\beta = \max\{\alpha_1, \alpha_2 \ldots \alpha_n\}$

Then $x \notin H_\mu$ for any $\mu > \nu$. Now there are two possibilities either

(i) $x \in H_\beta$ for some $\beta < \nu$ or (ii) Also $x \in H_\beta$ $\forall \beta < \nu$

and in this case, $x \in F_\beta \subseteq H_\beta$. Thus in either case, $x \in H_\beta$ for some $\beta < \nu$. Hence H forms an open covering of X.

Since $\bar{H}_\alpha \subseteq G_\alpha$, for each $\alpha \in I$ as shown above, it follows that H is a shrinking of C. i.e the given open covering C is shrinkable.

The converse part is the very same as for above theorem.

Exercise : Let β be a base for a normal T_1-space i.e for a T_4-space. Show that for each $G_i \in \beta$ and any point $p \in G_i$, $\exists$ a member $G_j \in \beta$ such that

$$p \in \bar{G}_j \subseteq G_i$$

Sol # $\because$ X is a T_1-space — singleton

$\therefore$ for every $p \in X$, $\{p\}$ is closed.

Now $p \in G_i \Rightarrow \{p\} \subseteq G_i$ i.e open set G_i contains a closed set $\{p\}$ — singleton

$\because$ X is normal

$\therefore$ By normality criterion $\exists$ an open set G s.t

$$\{p\} \subseteq G \subseteq \bar{G} \subseteq G_i$$

Since $p \in G$ and β is a base for X, $\exists$ a member (or set) G_j of β such that

$$p \in G_j \subseteq G \quad (\because G \text{ is union of members of } \beta)$$

Now $G_j \subseteq G \Rightarrow \bar{G}_j \subseteq \bar{G}$

Thus

$$p \in \bar{G}_j \subseteq \bar{G}$$

But $\bar{G} \subseteq G_i$

Hence $p \in \bar{G}_j \subseteq \bar{G} \subseteq G_i$

$$\Rightarrow p \in \bar{G}_j \subseteq G_i$$

(Proved)

Completely Normal Space & T_5- space

A top-space $(X.T)$ is completely normal if for each pair A, B of separated sets of X, there exist two disjoint open sets, one containing A and the other containing B

T_5- Space # A completely normal space which is also a T_1- space

Examples #

(i) An indiscrete space is completely normal.

(ii) The set of real number R with the left hand topology is completely normal

Theorem # Every completely normal space is normal and hence every T_5-space is T_4- space.

Proof # Let X be a completely normal space and F_1, F_2 be its two closed sets such that $F_1 \cap F_2 = \phi$

$\because$ Disjoint closed sets are separated.

$\therefore$ F_1 & F_2 are separated.

$\because$ X is completely normal

$\therefore$ $\exists$ two open sets U & V such that

$F_1 \subseteq U$, $F_2 \subseteq V$ & $U \cap V = \phi$

$\Rightarrow$ F_1 & F_2 are strongly separated

$\Rightarrow$ X is normal space.

Since T_5- space is a completely normal space, therefor it is normal space and is also a T_1-space and hence is a T_4- space.

Thus T_5-Space is a T_4- space.

Theorem # A top- space is completely normal iff every subspace of it is normal.

Proof # Let (X, τ) be a Completely normal space & Y a subset of X. Let F_1 & F_2 be two disjoint (relatively i.e. w.r.t relative topology) closed sets in (Y, τ_Y). Then $\exists$ two closed sets P_1 & Q_1 in (X, τ) such that

$F_1 = P_1 \cap Y \qquad F_2 = Q_1 \cap Y$

Here P_1 & Q_1 are not necessarily disjoint.

Now any two closed sets

$$P_1 \cap F_2 = [(P_1 \setminus F_1) \cup F_1] \cap F_2 \qquad P_1 = (P_1 - F_1) \cup F_1$$
$$= [(P_1 \setminus F_1) \cap F_2] \cup [F_1 \cap F_2]$$
$$= \phi \cup \phi = \phi$$

$(\because P_1 - F_1 \subseteq X - Y$ & $F_1 \subseteq Y$
$\therefore (P_1 - F_1) \cap F_2 = \phi$ as $F_2 \subseteq Y)$

Similarly $F_1 \cap Q_1 = \phi$

Now

$P_1 \cap F_2 = \phi$ & $F_1 \cap Q_1 = \phi$

$\Rightarrow F_2 \subseteq X - P_1 = P_1^c$ & $F_1 \subseteq X - Q_1 = Q_1^c$

$\Rightarrow F_1, F_2$ are weakly separated in X by open sets $X - Q_1$ & $X - P_1$.

$\because X$ is completely normal

$\therefore \exists$ open sets U & V such that

$F_1 \subseteq U \qquad F_2 \subseteq V \qquad U \cap V = \phi$

$\Rightarrow F_1 \cap Y \subseteq U \cap Y, \quad F_2 \cap Y \subseteq V \cap Y$

$\Rightarrow F_1 \subseteq U \cap Y \quad F_2 \subseteq V \cap Y \quad (\because F_1 \subseteq Y, F_2 \subseteq Y)$

Also

$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y$
$= \phi \cap Y = \phi$

$\Rightarrow F_1$ & F_2 are strongly separated by by the pair of two disjoint open sets $U \cap Y$ and $V \cap Y$. Hence (Y, τ_Y) is normal

OR

$F_1 \cap \bar{F}_{2X} = \bar{F}_{1Y} \cap F_2$ where $\bar{F}_{1Y}$ is closure

$= (Y \cap \bar{F}_{1X}) \cap \bar{F}_{2X}$ $\quad \because \bar{F}_{1Y} = Y \cap \bar{F}_{1X}$

$= Y \cap (\bar{F}_{1X} \cap \bar{F}_{2X})$

$= (Y \cap \bar{F}_{1X}) \cap (Y \cap \bar{F}_{2X})$

$= \bar{F}_{1Y} \cap \bar{F}_{2Y} = F_1 \cap F_2 = \phi$

$\Rightarrow F_1 \cap \bar{F}_{2X} = \phi$

Similarly $\bar{F}_{1X} \cap F_2 = \phi$

Hence F_1 & F_2 are separated subsets of X. Next proceed as above

Conversely let every subspace of a topological space (X, T) is normal. Let A & B be two weakly separated sets in (X, T). Then (by Hausdorff-Lennes)

$\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$

Let $G = X - (\bar{A} \cap \bar{B})$ $\quad (\because X - A = A^c)$

$= X - (\bar{A} \cap \bar{B}) = (\bar{A} \cap \bar{B})^c$

$= (\bar{A})^c \cup (\bar{B})^c = Ext(A) \cup Ext(B)$

Now $G \cap \bar{A}$ (closed) & $G \cap \bar{B}$ (closed) are closed in (G, T_G)

Also

$(G \cap \bar{A}) \cap (G \cap \bar{B}) = G \cap (\bar{A} \cap \bar{B})$

$= (\bar{A} \cap \bar{B})^c \cap (\bar{A} \cap \bar{B}) = \phi$ (Subspace)

$\Rightarrow G \cap \bar{A}$, $G \cap \bar{B}$ are disjoint, closed sets in (G, T_G)

$\because (G, T_G)$ is a normal space by Hypothesis

$\therefore$ sets $G \cap \bar{A}$, $G \cap \bar{B}$ can be strongly separated.

By pair of disjoint sets U, V open in (G, T_G).

Since G is open in X and U, V are open in (G, T_G), therefore there exists open sets U_1, V_1 in X such that

$U = G \cap U_1$ $\quad V = G \cap V_1$

But then U & V are open in (X, T) being $\cap$ of open sets

Also

$G \cap \bar{A} = [Ext(A) \cup Ext(B)] \cap \bar{A}$

$= [Ext(A) \cap \bar{A}] \cup [Ext(B) \cap \bar{A}]$

$= \phi \cup Ext(B) \cap \bar{A} = Ext(B) \cap \bar{A}$

$\supseteq Ext(B) \cap A \supseteq A \quad \because A \subseteq \bar{A}$

$\Rightarrow G \cap \bar{A} \supseteq A$

Similarly $G \cap \bar{B} \supseteq B$

Thus $A \subseteq G \cap \bar{A} \subseteq U$ and $B \subseteq G \cap \bar{B} \subseteq V$ i.e A & B are strongly separated by U & V in (X, T)

Hence (X, T) is completely normal.

OR

Example # (A normal space which is not completely normal)

Let $X = \{a, b, c, d\}$ and $T = \{\phi, \{d\}, \{b, d\}, \{c, d\}, \{b, c, d\}, X\}$.

Closed sets in T are

$X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$ and ϕ.

As no pair of (non-null) closed sets is disjoint, it follows that (X, T) is normal

The subset $Y = \{b, c, d\}$ with relative topology T_Y given by

$T_Y = \{X, \phi, \{d\}, \{b, d\}, \{c, d\}\}$ is not normal

Here closed sets are

$\phi, Y, \{c, b\}, \{b\}, \{d, c\}$

The pair of disjoint closed sets $\{b\}, \{c\}$ can not be strongly seprated. Thus (Y, T_Y) is not normal and Hence (X, T) is not completely normal.

Theorem # Every subspace of a Completely normal space is completely normal and hence every subspace of T_5-space is also T_5-space.

Proof # Let (X, T) be a completely normal space. Then every subspace of (X, T) is a normal space. Let (Y, T_Y) be a subspace of (X, T). Then every subspace of (Y, T_Y) is also a subspace of (X, T) and is therefore normal. Consequently every subspace of (Y, T_Y) is normal and hence (Y, T_Y) is Completely normal

because a top-space normal iff its every subspace is normal.

If (X, T) is in particular, a T_5-space i.e a Completely normal T_1-space, then (Y, T_Y) is also a Completely normal T_1-space i.e a T_5-space.

Separation By Real Functions

Let (X, T) be an aribitrary top-space, $C(X)$ or $C(X, R)$ be the set of all continuous real functions on X. (Ex: Two distinct points $x \neq y$). If for each pair of distinct points $x \neq y$ in X, there exists a function f in $C(X)$ such that

$$f(x) \neq f(y)$$

, we say that $C(X, R)$ separates points.

Two disjoint subsets $A \& B$ are said to be separated by a Continuous real function (or Completely separated) if there exists a function $f \in C(X)$, such that

$$f(x) = 0 \quad \forall x \in A \qquad f(x) = 1 \quad \forall x \in B$$

i.e $f(A) = 0$ $\qquad$ $f(B) = 1$

and $\quad 0 \leq f(x) \leq 1 \quad \forall x \in X$

i.e values of f lies in the closed interval

The function f is called the Urysohn function on X, Corresponding to the pair the pair of (disjoint) subsets $A \& B$

Theorem # In a top-space (X, T), if two subsets P, Q are separated by a continuous real function, then P, Q are also strongly separated (by a pair of disjoint opensets)

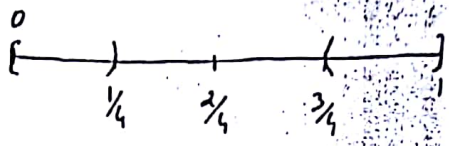
Proof ∵ Subsets P, Q are separated by a continuous real function

∴ Let f, denote the corresponding Urysohn function. Let $I = [0, 1]$ and σ_I be relative topology on I, relative usual topology σ on R, the set of real numbers. Then

$[0\ \ 1/4) = [0\ \ 1] \cap [0\ \ 1/4)$

and

$(3/4\ \ 1] = [0\ \ 1] \cap (3/4\ \ 1]$



$\Rightarrow [0\ \ 1/4)$ & $(3/4\ \ 1]$ are open sets in (I, σ_I)

∵ $f : X \to [0\ \ 1]$ is a continuous function

∴ $U = f^{-1}[0\ \ 1/4)$ & $V = f^{-1}(3/4\ \ 1]$ are open sets in (X, T)

Also

$$U \cap V = f^{-1}[0\ \ 1/4) \cap f^{-1}(3/4\ \ 1]$$

$$= f^{-1}\left([0\ \ 1/4) \cap (3/4\ \ 1]\right)$$

$$= f^{-1}(\phi) = \phi \qquad \because f^{-1}(\phi) = \phi$$

$\Rightarrow U$ & V are disjoint

∵ P & Q are separated by f.

∴ $f(P) = 0$ & $f(Q) = 1$

($f^{-1}(0) = P$ & $f^{-1}(1) = Q$.)

Now $0 \in [0\ \ 1/4)$ & $1 \in (3/4\ \ 1]$

$\Rightarrow \{0\} \subseteq [0\ \ 1/4)$ & $\{1\} \subseteq (3/4\ \ 1]$

$\Rightarrow f^{-1}\{0\} \subseteq f^{-1}[0\ \ 1/4)$ & $f^{-1}\{1\} \subseteq f^{-1}(3/4\ \ 1]$ $\left\{ A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B) \right.$

$\Rightarrow P \subseteq U$ & $Q \subseteq V$

Thus P & Q are strongly separated by U & V.

Theorem# If $C(X\,R)$ separates points of top-space $(X.T)$, then $(X.T)$ is Hausdorff space.

Proof# Let a, b be any two distinct points of X.

$\because C(X\,R)$ separates X

$\therefore \exists$ a function $f \in C(X\,R)$ such that $f(a) \neq f(b)$ (real valued function)

Now $f(a), f(b) \in R$, R is a Hausdorff space.

Therefore $\exists$ two open sets U & V such that

$f(a) \in U$ & $f(b) \in V$ $\quad U \cap V = \emptyset$

$\Rightarrow a \in \bar{f}^1(U)$ (open) & $b \in \bar{f}^1(V)$ (open) ——— (inverse image of open set is open)

$\because f$ is continuous

$\therefore \bar{f}^1(U)$ & $\bar{f}^1(V)$ are open sets in $(X\,T)$

Also

$$\bar{f}^1(U) \cap \bar{f}^1(V) = \bar{f}^1(U \cap V) \overset{\text{disjoint}}{=} \bar{f}^1(\emptyset) = \emptyset$$

$\Rightarrow a, b$ are strongly separated by open sets $\bar{f}^1(U)$ & $\bar{f}^1(V)$

Hence $(X.T)$ is a Hausdorff space.

Note T_1, T_2 and T_3 spaces cannot be described in terms of separation by Continuous real functions.

We will discuss normal spaces in terms of separation by continuous functions.

T_4 = normal

Theorem # (Urysohn's Lemma)

A top- space (X, T) is normall iff every pair of disjoint closed sets A, B in (X, T) are separated by a continuous real function i.e there exists a continuous real function f on (X, T) such that $f: X \to [0, 1]$.

$$f(A) = 0 \qquad f(B) = 1$$

and $0 \le f(x) \le 1 \quad \forall x \in X$

$0 \le f(x) \le 1$ bounded by 0 & 1

Proof # Let (X, T) be a topological space, such that there exists a continuous real function f on (X, T) satisfying the condition, for every paire of disjoint closed sets A, B i.e A, B are separated by a continuous function f. Then A, B are strongly separated by a pair of disjoint open sets in (X, T). Hence (X, T) is a normal space

Conversely, let (X, T) be a normal space and let A, B be a pair of disjoint closed sets in (X, T) i.e

$$A \cap B = \phi$$

$\Rightarrow A \subseteq B^c$ and B^c is an open set in X. (b/c B is close)

Hence by normality criterion, There exists an open set $U_{1/2}$ such that

$$A \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq B^c$$

Again $U_{1/2}$ is an open set containg a closed set A and B^c is an open set containg a closed set $\bar{U}_{1/2}$. So by normality criterion. There exist open sets $U_{1/4}$, $U_{3/4}$ such that

$$A \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{1/2} \quad \& \quad \bar{U}_{1/2} \subseteq U_{3/4} \subseteq \bar{U}_{3/4} \subseteq B^c$$

$$\Rightarrow A \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq U_{3/4} \subseteq \bar{U}_{3/4} \subseteq B^c$$

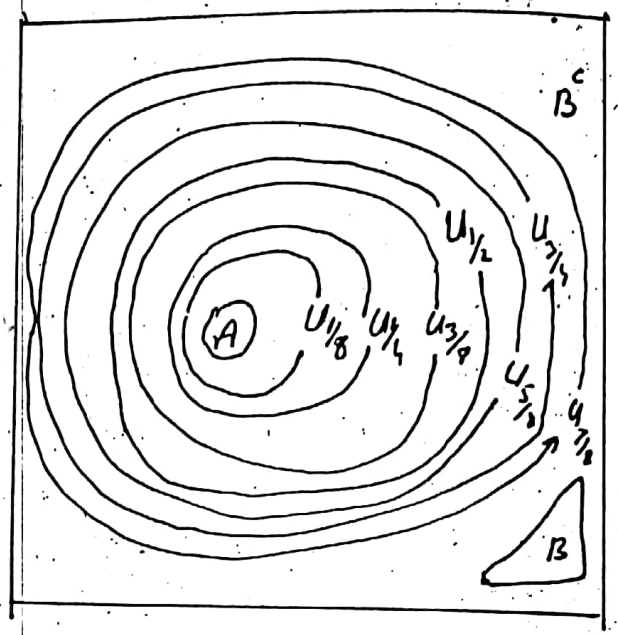
If we continue this process, for each dyadic rational number of the form $t = \frac{m}{2^n}$ $\quad n = 1, 2, \ldots$ $\quad m = 1, 2, \ldots, 2^n - 1$

, we have an open set of the form U_t, such that

$$t_1 < t_2 \Rightarrow$$

$$A \subseteq U_{t_1} \subseteq \bar{U}_{t_1} \subseteq U_{t_2} \subseteq \bar{U}_{t_2} \subseteq B^c$$

- dyadic No -



Explanation: In dyadic number $t = \frac{m}{2^n}$

Let $n = 2$ be fixed, then we have numbers $1/4$, $2/4 = 1/2$, $3/4$ and corresponding sets are $U_{1/2}, U_{1/4}, U_{3/4}$ & we have

$$A \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq U_{3/4} \subseteq \bar{U}_{3/4} \subseteq B^c$$

If $n = 3$ $\quad 2^3 = 8$ and we dyadic numbers

$1/8$, $2/8 = 1/4$, $3/8$, $4/8 = 1/2$, $5/8$, $6/8 = 3/4$, $7/8$

and we have

$$A \subseteq U_{1/8} \subseteq \bar{U}_{1/8} \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{3/8} \subseteq \bar{U}_{3/8} \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq U_{5/8} \subseteq \bar{U}_{5/8} \subseteq U_{3/4} \subseteq \bar{U}_{3/4}$$

$$\subseteq U_{7/8} \subseteq \bar{U}_{7/8} \subseteq B^c$$

The precedence is depicted in the fig above

We have for each dyadic fraction $\frac{m}{2^n}$, an open set $U_{\frac{m}{2^n}}$ with following properties.

(a) A is contained in each $U_{m/2^n}$

(b) each $\overline{U}_{m/2^n}$ is contained in B^c

(c) If $\frac{m}{2^n} < \frac{m'}{2^n}$, then $\overline{U}_{\frac{m}{2^n}} \subseteq U_{\frac{m'}{2^n}}$

now we define our function f as

$f(x) = 0$ if x is in every U_t $t = \frac{m}{2^n}$

$f(x) = \sup\{t : x \notin U_t\}$ i.e if x does not lie in any U_t

Then (i) for any x in X, $0 \le f(x) \le 1$, since $0 < \frac{m}{2^n} < 1$

(ii) for any x in A, x belongs to every $U_{\frac{m}{2^n}}$, so that

$f(x) = 0 \quad \forall x \in A$

i.e $f(A) = 0$

(iii) for any x in B, x does not lie in any $U_{\frac{m}{2^n}}$, so that $f(x) = 1 \quad \forall x \in B$ i.e

$f(B) = 1$

now we are to prove that f is continuous

All intervals of the form $[0\ a)$ & $(a\ 1]$, $0 < a < 1$ constitute an open subbase for $[0\ 1]$ because collection of all finite intersections of these form a base for $[0, 1]$.

To show that f is continuous, it is sufficient to show that $f^{-1}([0\ a))$ and $f^{-1}((a\ 1])$ are open.

$f(x) < a \iff x$ is in some U_t for $t < a$

$\Rightarrow f^{-1}([0\ a)) = \{x : f(x) < a\} = \bigcup_{t<a} U_t$ which is an open set. Similarly

$f(x) > a \iff x$ is outside of $\overline{U}_t$ for some $t > a$ and therefore

$f^{-1}((a\ 1]) = \{x : f(x) > a\} = \bigcup_{t>a} \overline{U}_{t'} = \overline{U}_t$

which is an open set

Note A function $f : (X, T_1) \to (Y, T_2)$ is continuous iff inverse image of members of the subbase of Y is open in X

OR

Urysohn's Lemma # A t$_0$-space X is normal iff for every two disjoint closed subsets A, B of X and closed interval $[a,b]$ of reals, there exists a continuous mapping $f: X \to [a,b]$ such that $f(A) = \{a\}$ & $f(B) = \{b\}$

Comments # In case of separable metric spaces, Urysohn found necessary and sufficient conditions for metrizability. The Urysohn Lemma is a basis for the proof

Proof # Suppose that for any pair of disjoint closed sets A, B and for any closed interval $[a,b]$ of reals, there exists a continuous mapping $f: X \to [a,b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$. Then for any pair of disjoint closed sets A, B and interval $[0,1]$, there exists a continuous mapping $f: X \to [0\ 1]$ such that $f(A) = 0$ & $f(B) = 1$. Hence X would be normal because

$$\bar{f}^1([0\ \tfrac{1}{2})) = A \quad \text{i.e} \quad A \subseteq \bar{f}^1([0, \tfrac{1}{2}))$$

and $\bar{f}^1((\tfrac{1}{2}\ 1]) = B$ i.e $B \subseteq \bar{f}^1((\tfrac{1}{2}, 1])$

$\because$ f is continuous and $[0\ \tfrac{1}{2})$ & $(\tfrac{1}{2}, 1]$ open in $[0, 1]$

$\therefore$ $\bar{f}^1([0, \tfrac{1}{2}))$ & $\bar{f}((\tfrac{1}{2}, 1])$ are open sets in X

Also

$$\bar{f}^1([0\ \tfrac{1}{2})) \cap \bar{f}^1[(\tfrac{1}{2}, 1]] = \bar{f}^1([0\ \tfrac{1}{2}) \cap (\tfrac{1}{2}, 1])$$

$$= \bar{f}^1(\phi)$$

$$= \phi$$

Thus A, B are separated by disjoint open sets $\bar{f}^1([0\ \tfrac{1}{2}))$ & $\bar{f}^1((\tfrac{1}{2}\ 1])$ and is so normal

Conversely suppose that X is a normal space we are to prove that for any pair of disjoint closed

A, B and for any interval [a, b], there exists cntu. a mapping $f: X \longrightarrow [a, b]$ such that

$f(A) = \{a\}$ & $f(B) = \{b\}$.

First we construct a mapping $g: X \to [0\ 1]$ such that

$g(A) = \{0\}$ & $g(B) = \{1\}$

We define a collection $\{G_r : r \text{ rational}\}$ of open sets such that

$\overline{G_r} \subseteq G_s$ if $r < s$ in the following way

Let $G_r = \phi$ for $r < 0$ and $G_r = X$ for $r \geq 1$

Let $G_1 = X - B = B^c$

$\because A \cap B = \phi \Rightarrow A \subseteq B^c = G_1$

$\Rightarrow G_1$ is an open set containg a closed set A.

By normality condition, there exists an open set G_0 containing A such that

$$A \subseteq G_0 \subseteq \overline{G_0} \subseteq G_1$$

Now let $\{r_n\}_{n \in N}$ be a collection of all rationals in [0 1] with $r_1 = 0$ $r_2 = 1$

We use induction for each $n \geq 3$. and seek an open set G_{r_n}.

Let r_i be the Largest rational and r_j is the smallest rational for $i, j < n$ such that

$$r_i < r_n < r_j$$

Then by normality criterion, we obtain the open set G_{r_n} such that

$$\overline{G_{r_i}} \subseteq G_{r_n} \text{ and } \overline{G_{r_n}} \subseteq G_{r_j}.$$

Now we define a mapping g as

$g(x) = \inf\{r : x \in G_r\}$

Since every $x \in X$ belongs to some G_r, but not to all of them, $g(x)$ is a well-defined real number in [0 1].

Also we note that

$g(x) < q$ iff $x \in G_r$ for some $r < q$

and c-

$$\{x : g(x) < q\} = \bigcup \{G_r : r < q\}$$

which is an open set.

Further $g(x) > q$ iff $x \notin \bar{G}_r$ for some $r > p$

and so

$$\{x : g(x) > q\} = \bigcup \{X \setminus \bar{G}_r : r > p\}$$

which is an open set

Thus $g^{-1}((p, q))$ is always an open set in X. Hence g is continuous

Clearly for each $x \in A$, we have some set G_r such that $x \in G_r$. So

$$g(A) = \inf\{r : x \in G_r\}$$
$$= \{0\}$$

Also for each $x \notin A$, $x \notin \bar{G}_r$ for any r i.e

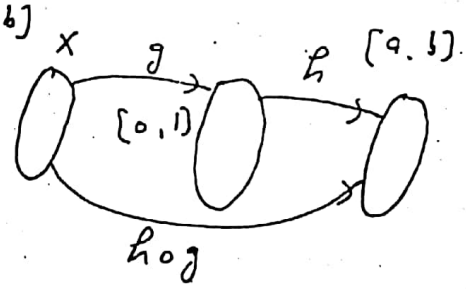
$g(x) = 1 \quad \forall x \in B$

$\Rightarrow g(B) = 1$

Now defined a function $h(x)$ as : $[0,1] \longrightarrow [a, b]$ as

$h(x) = (b-a)x + a$, then h is continuous

Then mapping $h \circ g : X \to [a, b]$ is continuous being the product of two continuous functions and



$$(h \circ g)(A) = h[g(A)]$$
$$= h(0) = \{a\}$$
$$(h \circ g)(B) = h[g(B)]$$
$$= h(1) = \{b\}$$

Thus $f = h \circ g$ is the required mapping

Completely Regular Spaces

A top-space (X, T) if for any closed set F, any point x lying outside (not in) F, there always exists a continuous real function $f: X \to [0, 1]$ such that

$$f(x) = 0 \qquad f(F) = 1$$

i.e $\{x\}$ and F are separated by a continuous real function f on (X, T)

($x \notin F$; separated by open set; Regular Sp.; by Urysohn function and)

Since constant functions are continuous, the above condition is equivalent to the existence of a continuous function $g: X \to [0, 1]$ such that

$$g(F) = 0 \qquad g(x) = 1$$

($f(x) = 0$, $f(F) = 1$ by interchanging; constant function which is conti...)

because if we take

$$g = I - f \quad \text{where } I: X \to [0, 1] \text{ is}$$

given by $I(x) = 1 \quad \forall x \in X$, then

$$g(F) = (I - f)(A) = I(A) - f(A) = 1 - 1 = 0$$

and $g(x) = (I - f)(x) = I(x) - f(x)$

$$= 1 - 0 = 1$$

A continuous function f which satisfies conditions given in the above definition is said to separate the point x and the set F

Tychonoff (Tichonov) Space

A completely regular space which is also a T_1- space is called Tychonoff space

Theorem # A completely regular space is regular space.

Proof # Let X be a completely regular space

F & x are separated

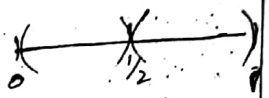
, F be a closed set and x a point not in F.

$\because (X, T)$ is completely regular

$\therefore$ there is a continuous function $f: X \to [0, 1]$ such that

$$f(x) = 0 \quad \text{and} \quad f(F) = 1$$ (due to comp. regular)

Let $U = [0, 1/2)$, $V = (1/2, 1]$. Then U & V are open sets in $[0\ 1]$ & because f is continuous, $f^{-1}(U), f^{-1}(V)$ are open in X.



Also

$x \in f^{-1}(U)$ $\quad A \subseteq f^{-1}(V)$

and $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$

$= f^{-1}(\phi)$

$= \phi$

Thus X is a regular space.

$\{0\} \subseteq (0, \frac{1}{2})$

$f^{-1}\{0\} \subseteq f^{-1}(U)$

$\Rightarrow x \in f^{-1}(U)$.

$\{1\} \subseteq (\frac{1}{2}, 1]$

$f^{-1}(\{1\}) \subseteq f^{-1}(V)$

$F \subseteq f^{-1}(V)$.

$A \subseteq B$

$f(A) \subseteq f(B)$

$f^{-1}(A) \subseteq f^{-1}(B)$

Example # Every metric space is completely regular, in fact a tychonoff space

Let (X, d) be a metric space, A a closed set in X, $x \notin A$.

We define a function $f: X \to [0\ 1]$ by

$$f(y) = d(y, A), \quad y \in X$$

Then f is continuous because for any $\epsilon > 0$, $\forall\ y, y' \in X$

$$d(y, y') < \epsilon = \delta$$

$$\Rightarrow |f(y), f(y')| = |d(y, A) - d(y', A)| < \epsilon$$

$$\Rightarrow d(y, y') < \delta \Rightarrow |f(y) - f(y')| < \epsilon$$

Also $f(A) = 0$

and for any x not in A

$$f(x) = d(x, A) = k \neq 0$$

because A is closed and $x \notin A$ ($d(x, A) \neq 0$ if $x \in A$, x is limit pt of A)

Taking $g = (\frac{1}{k}) f$, then g is continuous and has the required properties

Since every metric space is T_1-space, so X is a Tychonoff space

completely regular
C.R. + T_1

Theorem # (i) Every T_4-space is a Tychonoff space
(ii) Every Tychonoff space is completely regular.
(iii) Every Tychonoff space is a T_3-space.

Normal + T_1 = T_4
Regular + T_1 = T_3

Proof # (i) Let $(X, \mathcal{T})$ be a T_4-space and F be a closed set and x any point not in F, $x \neq F$ in $(X, \mathcal{T})$.

$\because$ $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

$\therefore$ $(X, \mathcal{T})$ is T_1-space

$\{x\} \cap F = \phi$ disjoint

$\Rightarrow$ Every singleton will be a closed set

$\Rightarrow$ $\{x\}$ is closed set disjoint with F

$\because$ T_4 is a normal T_1-space

$\therefore$ By Urosohn Lemma there exist a continuous real function $f : X \longrightarrow [0\ 1]$ such that

$f(x) = 0$ (disjoint pts.) $\quad f(F) = 1$ (closed set) or $f(x) = 1 \quad \forall x \in F$

Hence $(X, \mathcal{T})$ is a completely regular space and hence also a tychnoff space because it is T_1-space

(ii) Since a tychonoff space is a completely regular space which is also a T_1-space, every tychonoff space is completely regular space

(iii) We know that a regular T_1-space is T_3-space and a Tychonoff space is completely regular T_1-space.

Also we know that every completely regular space is regular

Hence every Tychonoff space is T_3-space

Note By virtue of properties (i) & (iii), a Tychonoff space is also called a $T_{3\frac{1}{2}}$-space

Since a normal space need not be regular, it also need not be completely regular. However the following implication does hold.

Theorem A normal space is completely regular iff it is regular

Proof Let (X, T) be a normal space. If it is completely regular, then it is necessarily regular because a completely regular space is also regular

Conversely let (X, T) both regular and normal. Let F be a closed subset of X not containing the point x. Then $X - F = F^c$ is an open set containing point x.

By regularity Criterion there exist an open set V in X such that

$$x \in V, \quad \bar{V} \subseteq X \setminus F = (F^c) \quad \bar{V} \cap F = \emptyset$$

$\because$ F and $\bar{V}$ are disjoint closed sets in the normal space

$\therefore$ By Urysohn lemma there exists a continuous function $f : X \to [0, 1]$ (Urysohn function) such that

$f(F) = 1$ & $f(\bar{V}) = 0$

$\because$ $x \in \bar{V}$ $\therefore$ $f(x) = 0$ Also $f(F) = 1$

So X is completely regular space.

Theorem A top space (X, T) is completely regular iff, for each point p in X and each member U belonging to a subbase S of T, such that $p \in U$, there exists a continuous function f on (X, T) such that

$f(p) = 0$ $f(x) = 1$ if $x \in X - U$ and $0 \le f(x) \le 1$ $\forall x \in X$

i.e for every subbasic member U containing p, there exists a function f on (X, T) such that

$f(p) = 0$ $f(U^c) = 1$ $0 \le f(x) \le 1$ $\forall x \in X$

i.e $\exists$ a Urysohn function such that

$f(p) = 0$ $f(U^c) = 1$ $0 \le f(x) \le 1$ $\forall x \in X$

Let β be a base generated by S. $\therefore p \in (p^c)$

Proof# Let for every subbasic set U containing p, there exist a continuous function $f: X \to [0\ 1]$ s.t

$f(p) = 0$ $\quad f(U^c) = 1$

Let F be any closed set not containing p. Then $p \in X - F = F^c$.

Let β be a base generated by subbase S. Since F^c is an open set containing p, it will be a super set of a basic set containing p. Let V be an open set, belonging to β such that

$$p \in V \subseteq X - F = F^c$$

$\because$ S generates β and $V \in \beta$

$\therefore$ V is an intersection of finitely many members of the subbase S, say

$$V = U_1 \cap U_2 \cap \cdots \cap U_n \text{, where } U_i \in S \text{ for } i = 1, 2 \cdots n$$

$\Rightarrow p \in U_i \quad \forall i = 1, 2 \cdots n$

By Hypothesis there exists, a continuous real function.

$f_i : X \to [0\ 1]$, such that

$$f_i(p) = 0 \qquad f_i(x) = 1 \quad \forall x \in X - U_i$$

and this is true for $i = 1, 2 \cdots n$

Let $g(x) = \sup\{ f_i(x) : i = 1, 2 \cdots n \}$

Then g is a continuous real function on (X, T), such that $g : X \to [0, 1]$

$$0 \le g(x) \le 1 \qquad \forall x \in X.$$

Also $g(p) = 0$. $\quad \{ g(p) = \sup\{ f_1(p), f_2(p), \cdots f_n(p) \} = 0$

and since $V \subseteq F^c$, $\quad V \cap F = \phi$

$$\Rightarrow F \subseteq V^c = X \setminus V$$

for $x \in F \subseteq X \setminus V$ $\quad x \in X - U_j$ for some $j = 1, 2 \cdots n$

because

$$V = U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_n$$

$$\Rightarrow V^c = U_1^c \cup U_2^c \cup U_3^c \cup \cdots \cup U_n^c$$

Consequently

$$g(x) = 1 \qquad \forall x \in F$$

Hence (X, T) is completely regular space

Conversely let (X,T) be a completely regular space and $p \in U$ for some $U \in S$ (S being a subbase of T). Then $X-U = U^c$ is a closed set not containing p. Hence by complete regularity of (X,T), $\exists$ a continuous real function $f: X \to [0\ 1]$ such that

$$f(p) = 0 \qquad f(x) = 1 \quad \forall x \in X \setminus U$$

Theorem # Every subspace of a completely regular space (or of a Tychonoff space) is completely regular (is Tychonoff) OR To be completely regular is a hereditry property.

Proof # Let (A, T_A) be a subspace of a completely regular space (X,T). Let F be a closed set in (A,T_A) and p be a point in A not in F.

$\because$ F is relatively closed set.

$\therefore$ $\exists$ a closed set K in X such that

$$F = A \cap K.$$

$\because$ $p \notin F \Rightarrow p \notin K$

$\Rightarrow$ K is a closed set in (X,T) not containing p

Since (X,T) is completely regular, therefore there is a Urysohn function $f: X \to [0\ 1]$ such that.

$$f(p) = 0 \qquad f(K) = 1 \quad \text{or } f(x) = 1 \quad \forall x \in K$$

Let $h = f|_A$ (restriction of f on A), then h is also continuous function on (A, T_A) because restriction of a continuous function is also continuous

Also.

$$h(p) = 0 \qquad h(x) = 1 \quad \text{if } x \in F = A \cap K$$

and. $0 \leq h(x) \leq 1 \quad \forall x \in A$.

Hence the subspace (A, T_A) is also completely regular.

Every subspace of Tychonoff is Tychonoff because every subspace of completely regular space is completely regular and every subspace of T_1-space is also a T_1-space.

Theorem # If x and y are two distinct points in a Tichonov space X, then there exists a real valued continuous mapping f of X such that

$$f(x) \neq f(y)$$

Proof # Let X be a Tichnov space and x, y be two distinct points

$\because$ X is T_1 - space and is also completely regular

$\therefore$ $\{y\}$ is a closed set not containg x

By complete regularity of X, there exists a function $f : X \longrightarrow [0\ 1]$ such that

$$f(x) = 0 \qquad f(\{y\}) = \{1\}$$

$\Rightarrow f(x) \neq f(y)$

Cozero Set of a function

If f is a real valued continuous mapping of the topological space X, the set all points of X at which f is not equal to zero, is called cozero set of f.

Relation b/w Collection of all real Valued functions on a Tichonov Space and its Topology

We show a very close relationship b/w the collection of real valued continuous functions on a Tichonov space and its topology by following Theorem.

Theorem # A Hausdorff space $(X, \mathcal{T})$ is completely regular (Tichonov) iff the family of all cozero sets of real valued continuous mappings of X is a base for the topology T on X.

Proof # Let $(X, \mathcal{T})$ be a Tichonove space and $x^* \in G^* \in T$.

Hausdorff space = separated

Then $x^* \notin X \setminus G^*$

So by Complete regularity of X, there exists a real valued continuous mapping f of X such that

$$f(x^*) = 0 \quad \& \quad f(X - G^*) = \{1\}$$

Let $g(x) = 1 - f(x) \quad \forall x \in X$

we get a continuous function g of X such that

$$g(x^*) = 1 - f(x^*) = 1 - 0 = 1 \quad \text{and} \quad g(X \setminus G^*) = 1 - f(X - G^*) = \{0\}$$

Thus the cozero set of g contains x^* and is a subset of G^*, Since each cozero set is open and their union is X, this family forms a base for $\mathcal{T}$

Conversely suppose that the family of cozero sets of all mapping of X is a base for $\mathcal{T}$. Let F be any closed set and p is a point not in F.

Then $p \notin X \setminus F = F^c$

$\because F^c$ is open set and family of cozero sets form a base for $\mathcal{T}$

$\therefore$ there must be a cozero set E of some real valued mapping f of X such that

$$p \in E \subseteq X \setminus F = F^c$$

Thus $f(p) \neq 0$.

Define a mapping h as

$$h(x) = \max\{0, \min\{1, 1 - f(x)/f(p)\}\} \quad \forall x \in X$$

Clearly

$$h(p) = \max\{0, \min\{1, 1 - \tfrac{f(p)}{f(p)}\}\} = 0$$

and $h(F) = 1$

Also h is a continuous function of X

Thus $(X, \mathcal{T})$ is completely regular

Theorem # The topological product $(X, \mathcal{T})$ of any collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}$ is completely regular iff each of the spaces $\{X_\alpha, \mathcal{T}_\alpha\}$ is completely regular

P.T.O

Proof # Let each of the spaces $\{X_\alpha : \alpha \in I\}$ be completely regular.

Let $X = \prod_{\alpha \in I} X_\alpha$ be the topological product

Let $x = \{x_\alpha\}$ be a point in X, and U a member of sub-base S, which generates the product topology T on X such that $x \in U$. Then.

$$U = P_\beta^{-1}(U_\beta) \quad \text{for some } U_\beta \in T_\beta, \text{ and for some } \beta \in I$$

where P_β is the projection mapping of X onto X_β.

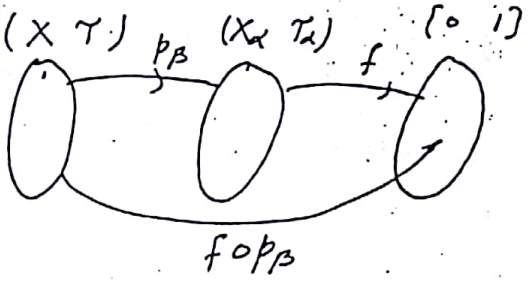
Also $x_\beta = P_\beta(x) \in U_\beta$.

As $(X_\beta\ T_\beta)$ is completely regular, there exists a continuous real function f on $(X_\beta\ T_\beta)$ such that

$$f(x_\beta) = 0 \qquad f(y) = 1 \quad \forall\ y \in X_\beta - U_\beta$$

and $0 \le f(y) \le 1 \quad \forall\ y \in X$

Then $g = f \circ P_\beta$ being the product of two continuous functions is continuous on $(X\ T)$



and

$$g(x) = (f \circ p_\beta)(x)$$
$$= f[P_\beta(x)]$$
$$= f(x_\beta) = 0$$

$$g(y) = f(P_\beta(y)) \qquad \forall\ y \in X - U = X - P_\beta^{-1}(U_\beta)$$
$$= 1 \qquad \because P_\beta(y) \in X_\beta - U_\beta$$

Also

$$0 \le g(y) \le 1 \qquad \forall\ y \in X$$

Hence by a previous theorem $(X\ T)$ is a completely regular

OR

Let $x = \{x_\alpha\} \in X$ and F be a closed subset of X not containing x. Then there is an index α_i say such that for any $y = \{y_\alpha\}$ in F

$$x_{\alpha_i} \neq y_{\alpha_i} \qquad x_{\alpha_i},\ y_{\alpha_i} \in X_{\alpha_i}$$

Choose an open set U_{α_i} in X_{α_i} which contains no y_{α_i} for any $y = \{y_{\alpha_i}\}$.

Then $y_{\alpha_i} \in X_{\alpha_i} \setminus U_{\alpha_i} = F_{\alpha_i}$, a closed subset of X_{α_i}.

Since X_{α_i} is completely regular, there is a function $f_{\alpha_i} : X_{\alpha_i} \longrightarrow [0\ 1]$ such that

$$f_{\alpha_i}(x_{\alpha_i}) = 0 \qquad f_{\alpha_i}(F_{\alpha_i}) = 1$$

Let $\pi_{\alpha_i} : X \longrightarrow X_{\alpha_i}$ denote the projection function and

$$f = f_{\alpha_i} \circ \pi_{\alpha_i}$$

Then f is continues function on (X, T) and.

$$f(x) = 0 \qquad f(F) = 1$$

So X is completely regular space.

Conversely let (X, T) be completely regular. Then every subspace of it is completely regular. Since each (X_α, T_α) is homeomorphic with a subspace of $(X\ T)$, each (X_α, T_α) is also completely regular.

Since the topological product of any collection of T_1-Spaces is a T_1-space and a subspace of T_1-space is T_1-, top-(space) product of topological spaces is a Tychonoff space iff each of spaces is a Tychonoff space.

* Functions on T_i-Spaces *

Let $f : (X, T) \longrightarrow (Y, T')$ be a function. We explore certain of (X, T) which remain invariant under different types of functions f on it.

Theorem# Let $f:(X,\mathcal{T}) \longrightarrow (Y,\mathcal{T}')$ be a function. Then

(i) f is a closed surjection and $(X,\mathcal{T})$ is T_1-space, then $(Y,\mathcal{T}')$ is also a T_1-space i.e T_1-property remains invariant under closed surjection

(ii) If f is closed bijection and $(X,\mathcal{T})$ is a T_2-space, then $(Y,\mathcal{T}')$ is also a T_2-space i.e T_2-property is invariant under a closed bijective function

(iii) if f is closed continuous function, surjective and $(X,\mathcal{T})$ is T_i, then $(Y,\mathcal{T}')$ is also a T_i-space for $i=4,5$ i.e T_4, T_5 properties remain invariant under closed continuous surjective function.

Proof# (i) Let $(X,\mathcal{T})$ be a T_1-space, and p be any point in Y

$\because$ f is surjective

$\therefore$ $\exists$ a point x in X such that $f(x) = p$

Since X is T_1, one pointic set $\{x\}$ is closed in X.

As f is a closed mapping, the one pointic set $\{p\} = \{f(x)\}$ is closed in $(Y,\mathcal{T}')$. Consequently $(Y,\mathcal{T}')$ is T_1-space.

(ii) # Let $(X,\mathcal{T})$ be T_2-space and f a closed bijection on $(X,\mathcal{T})$ onto $(Y,\mathcal{T}')$, Then f is also an open mapping because a mapping is open iff it is also a closed mapping

Let p, q be a pair of distinct points in Y. Then $f^{-1}(p)$, $f^{-1}(q)$ are two distinct points in $(X,\mathcal{T})$, since f is bijective. As $(X,\mathcal{T})$ is a T_2-space, there exist two disjoint open sets G, H in X such that

$f^{-1}(p) \in G$ and $f^{-1}(q) \in H$

$\Rightarrow$ $p \in f(G)$ and $q \in f(H)$

$\because$ f is bijective open mapping

$\therefore$ $f[G]$ & $f[H]$ are open in Y

Also $f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$
Thus $f(G)$ & $f(H)$ separate P & Q strongly.
Hence (Y, τ') is a T_2-space

(iii) Let f be a closed, continuous surjection of T_4-space (or a T_5-space) onto (Y, τ')

As (X, τ) is a T_1-space and f is a closed surjection, (Y, τ') is also a T_1-space.

Let P, Q be two disjoint (respectively weakly separated) in (X, τ). Then $f^{-1}(P)$ and $f^{-1}(Q)$ are are two disjoint closed (respectively, weakly separated) subsets in (X, τ), since f is a continuous surjection. Since X is normal (respectively Completely normal), there exist two disjoint open sets G, H such that

$$f^{-1}(P) \subseteq G \quad \text{and} \quad f^{-1}(Q) \subseteq H$$

$A \subseteq B \Rightarrow B^c \subseteq A^c$

$\Rightarrow X - G^{cc} \subseteq X - f^{-1}(P)^{c} \;\&\; X - H^{=Hc} \subseteq X - f^{-1}(Q)$

$f(x) = Y$, $x = f^{-1}(Y)$

$\Rightarrow X - G \subseteq f^{-1}(Y) - f^{-1}(P)$ By Complementing in X & $X - H \subseteq f^{-1}(Y) - f^{-1}(Q)$

$\Rightarrow X - G \subseteq f^{-1}(Y - P)$ & $X - H \subseteq f^{-1}(Y - Q)$ $\because f^{-1}(A - B) = f^{-1}(A) \setminus f^{-1}(B)$

$\Rightarrow f(X - G) \subseteq f f^{-1}(Y - P)$ & $f(X - H) \subseteq f f^{-1}(Y - Q)$ if $A \subseteq B$, $f(A) \subseteq f(B)$

Apply $\Rightarrow f(X - G) \subseteq Y - P$ & $f(X - H) \subseteq Y - Q$

Complementing in Y we have.

$P \subseteq Y - f(X - G)$ & $Q \subseteq Y - f(X - H)$

$\because f$ is closed.

$\therefore f(X - G)$ & $f(X - H)$ are closed in in Y

$\Rightarrow U = Y - f(X - G)$ & $V = Y - f(X - H)$ are open set in Y (complement of closed $\Rightarrow$ U & V are open)

Also $U \cap V = \{Y - f(X - G)\} \cap \{Y - f(X - H)\}$

$= Y \setminus \{f(X - G) \cup f(X - H)\}$

$= Y \setminus f(G^c \cup H^c) = Y \setminus f(X)$ ($G \cap H = \phi \Rightarrow G^c \cup H^c = X$)

$= Y \setminus Y = \phi$ $\because f$ is surjective.

Thus $P \notin Q$ are strongly separated by $U \notin V$.
Hence $(Y, \bar{T}')$ is normal (respectively completely normal) space and Thus is T_4-space (respectively T_5-space).

Results If $f: X \to Y$ and $\{A_\alpha : \alpha \in I\}$ be a family of subsets of X, then

(i) $f(\cup\{A_\alpha : \alpha \in I\}) = \cup\{f(A_\alpha) : \alpha \in I\}$

(ii) $f(\cap\{A_\alpha : \alpha \in I\}) \subseteq \cap\{f(A_\alpha) : \alpha \in I\}$ where equality need not hold

(iii) $A \subseteq B \Rightarrow f(A) \subseteq f(B)$

(iv) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, where equality need not hold

(v) $f(E) = \phi$ iff $E = \phi$

(vi) no general containment holds between
$f(X - E) = f(C_X E)$ & $Y - f(E)$
However a mapping $f: X \to Y$ is onto iff
$f(X - E) \supseteq Y - f(E)$

(i) $\bar{f}^1(\cup\{B_\alpha : \alpha \in I\}) = \cup\{\bar{f}^1(B_\alpha) : \alpha \in I\}$ $B_\alpha \subseteq Y$

(ii) $\bar{f}^1(\cap\{B_\alpha : \alpha \in I\}) = \cap\{\bar{f}^1(B_\alpha) : \alpha \in I\}$

(iii) if $S \subseteq T$, then $\bar{f}^1(S) \subseteq \bar{f}^1(T)$

(iv) $\bar{f}^1(S \setminus T) = \bar{f}^1(S) - \bar{f}^1(T)$

(v) $\bar{f}^1(S) = \phi$ iff $S \cap f(X) = \phi$

(vi) $\bar{f}^1(Y - S) = X - \bar{f}^1(S)$

(vii) $\bar{f}^1(\phi) = \phi$ and $\bar{f}^1(Y) = X$

(viii) $M \cap N = \phi$ in $Y \Rightarrow \bar{f}^1(M) \cap \bar{f}^1(N) = \phi$ in X

(ix) $A \subseteq \bar{f}^1(M) \Rightarrow f(A) \subseteq M$ where $A \subset X$, $M \subseteq Y$

Theorem # A topological space (Y, τ'), homeomorphic to a T_i-space (X, τ) is also a T_i-space, for $i = 0, 1, 2, 3, 4$ and 5

In other words, the property "that a topological space (X, τ) is a T_i-space" is a topological property for $i = 0, 1, 2, 3, 4$ & 5

Proof # Let f be a homeomorphism of (X, τ) onto (Y, τ') and p, q be a pair of distinct points in Y and X is T_0

$\because$ f is homeomorphism

$\therefore$ $\bar{f}^1$ is also homeomorphism and so injective

$\Rightarrow$ $\bar{f}^1(p)$ & $\bar{f}^1(q)$ are distinct points in (X, τ)

$\Rightarrow$ $\exists$ an open set G_1 which contains only one of the points $\bar{f}^1(p)$, $\bar{f}^1(q)$ but not the other.

Let $\bar{f}^1(p) \in G_1$ & $\bar{f}^1(q) \notin G_1$

$\Rightarrow$ $p \in f(G_1)$ & $q \notin f(G_1)$

$\because$ f is Homeomorphism $\therefore$ f is open

$\Rightarrow$ $f(G_1)$ is an open set in y which contains p but not q. Hence (Y, τ') is also T_0-space.

Now let (X, τ) be a T_1-space.

Then $\bar{f}^1(p)$, $\bar{f}^1(q)$ will be distinct points in X.

$\because$ X is T_1-space

$\therefore$ $\exists$ open sets U & V such that

$\bar{f}^1(p) \in U$. $\bar{f}^1(q) \notin U$

$\bar{f}^1(q) \in V$ $\bar{f}^1(p) \notin V$

$\Rightarrow p \in f(U)$ $q \notin f(U)$

$q \in f(V)$ $p \notin f(V)$

Thus $f(U)$, $f(V)$ are open sets in y which separate p, q weakly.

Hence (Y, τ') is also a T_1-space.

(ii) Let p be a point & Q a closed set not containing the point p

$\because$ f being a homeomorphism is a closed set

$\therefore$ $f(Q)$ is closed set and $f(p)$ is a point

$\because$ f is one-one

$\therefore$ $f(p) \notin f(Q)$

Now let $(X\ T)$

Summary of the CHAPTER

(1) Weak Separation

Two non-empty subsets A, B of (X, T) are weakly separated if $\exists$ two open sets G & H s.t

$A \subseteq G$ $\quad B \subseteq H$ $\quad A \cap H = \phi$ & $B \cap G = \phi$

necessary and sufficient condition are

(i) $A \cap B = \phi$ $\quad A \cap B^d = \phi$ $\quad B \cap A^d = \phi$

(ii) $(\bar{A} \cap B) \cap (B \cap \bar{A}) = \phi$ (Hausdorff Lennes Condition)

or equivalently

$A \cap \bar{B} = \phi$ $\quad B \cap \bar{A} = \phi$

(2) Strong Separation

Two non-empty subset A, B of (X, T) are strongly separated if $\exists$ two open sets G & H such that

$A \subseteq G$ $\quad B \subseteq H$ $\quad G \cap H = \phi$

* Two separated sets are always disjoint but converse may not be true

☆ Subsets of two weakly/strongly separated sets are also weakly/strongly separated

☆: Two open (closed) subsets of a top (X, T) are separated iff they are disjoint

* If the union of two separated sets is an open (closed) set, then the individual sets are open (closed)

*(3) T_0 - space

A top space (X, T) is T_0-space if for every pair of distinct points in X $\exists$ an open set which contains only one of the points but not the other

* Subspace of a T_0-space is T_0-space

* A top-space (X, T) is T_0-space iff

(i) for every pair pair of distinct ponts p, q in X

$p \notin \{\overline{q}\}$ or $q \notin \{\overline{p}\}$

(ii) Closures of distinct points are distinct.
(iii) for every $p \in X$, $\{p\}^d$ is a union of closed sets

* if $f:(X,T_1) \to (Y,T_2)$ is continuous and (Y,T_2) is a T_0-space, then f is necessarily constant.

4) T_1- Space #

A top-space (X,T) is T_1-space if every pair of distinct points is weakly separated.

* Every discrete space is T_1-space
* A top-space is T_1-space iff
  (i) every one pointic set is closed
  (ii) Every finite set is closed in X
  (iii) for every point $x \in X$, $\{x\}^d = \phi$ i.e derived set of every singleton set is nullset
  (IV) For every subset $A \subseteq X$, any limit point p of A every open set containg p intersects A in a denumberably infinite collection of points.
* In a T_1-space X, A point x is a limit point of a set A iff every open set contains x contains an infinite no of distinct points of A
* In a a T_1-space no finite subset has a limit point
* The coarsest T_1-topology that can be defined on a set X, is the co-finite topology
* The intersection of any collection of T_1-topologies on a set X is also a T_1-topology
* If (X,T_i) $i = 0,1,2$ is a T_i-space, then any finer topology of T_i is also T_i

T_2- Space # (Hausdorff space)

A top-space is T_2-space if every pair of distinct points is strongly separated by two disjoint open sets.

* Every T_2-space is T_1-space but the converse may not be true

* A top-space X is T_2-space iff diagonal of $X \times X$ is closed in $X \times X$
* Every subspace of a Hausdorff space is Hausdorff space
* A top-space (X, T) is a Hausdorff space iff for any two distinct points $a, b \in X$, there are closed sets F_1 & F_2 such that
(i) $X = F_1 \cup F_2$ (ii) $a \in F_1, b \notin F_1$ $b \in F_2, a \notin F_2$
* any Convergent sequence in a Hausdorff space has a unique limit
* The Homomorphic image of a Hausdorff space is a Hausdorff.
* The topological product of Hausdorff spaces is a Hausdorff space.

Regular Space

A top space X is regular if any closet set F and any point not in F are strongly separated
* Regularity Criterion: A top space X is regular iff for any point $x \in X$ and any open set U (or nbd) of x, there an open set V (or a nbd) such that $\bar{V} \subseteq U$
* A top-space X is regular iff
(i) the closed nbds of any point $x \in X$ form a base at x
(ii) For every closed subset A, the intersection of all closed nbds of A in (X, T) is the set A
* The closures of two distinct singletons are either identical or disjoint
* Every subspace of a regular space is regular

T_3 – Space

A top-space which is both regular and T_1 is called T_3-space.
* Definitions of regular space and T_3-space can be interchanged.

* any T_3-space is T_2-space
* Every regular T_0 space is T_1 space and hence T_3-space
* In T_3-space each pair of distinct points have nbds whose closures are disjoint

Normal Space

A top-space X is normal iff each pair of disjoint closed sets is strongly separated.

* Normality Criterion: A top-space X is normal iff for every closed set F and any open set U containing F $\exists$ an open set V such that

$$F \subseteq V \subseteq \bar{V} \subseteq U$$

* A top-space is normal iff each pair of disjoint closed sets have nbds whose closures are disjoint
* Every discrete space with at least two points is normal
* Every metric space is normal
* A closed subspace of a normal space is normal
* Every regular Lindeloff space is normal
* A closed continuous image of a normal space is normal

T_4-Space

A normal T_1-space is called T_4-space

* A T_4-space is regular space

Completely Normal Space

A top-space (X, T) is Completely normal space if for each pair of separated sets is (Completely) strongly separated.

T_5-Space

A Completely normal space which also a T_1-space

* A top- space is completely normal iff every subspace of it is normal
* Every subspace of a completely normal space is completely normal

Separation By Real functions

Two disjoint subsets A & B are said to separated by a continuous real function. (or completely seperated) if there exists a function $f \in C(X)$ such that

$$f(A) = 0 \qquad f(B) = 1$$

and $\quad 0 \leq f(x) \leq 1 \qquad \forall x \in X$

* In a top- space if two subsets P, Q are separated by a continuous real function, then P, Q are also separated strongly separated by a pair of disjoint open sets.
* If $C(X, R)$ separates points of top- space (X, T), then (X, T) is Hausdorff space.
* A top- space is normal iff every pair of disjoint closed sets is separated by Urysohn function.

Completely Regular Space

* A top- space is said to be completely regular space if any closed set and any point outside the closed set are separated by a Urysohn's function.
* A completely regular space is regular
* Every metric space is completely regular
*

Tychonoff Space

A completely regular space which is also a T_1-space is called Tychonoff space.

* Every T_4- space is a Tychonoff space
* Every T_4- space is completely regular
* Every Tychonoff space is T_3- space

* A normal regular space is Completely regular.
* A top- space (X, T) is completely regular iff for each point $p \in X$ and every subbasic open set U containing p are separated by Urysohn function
* Every subspace of a completely regular space is completely regular space.
* The top- product of any collection of topological spaces is completely regular iff each of the spaces is completely regular

Functions on T_i - Space

* T_1- property is conserved under closed surjection. i.e if $f: X \to Y$ is closed surjection and X is T_1-space, then Y is also T_1-space
* T_2- property is invariant under closed bijection.
* T_4, T_5 are invariant under closed continuous surjective function.
* $T_0, T_1, T_2, T_3, T_4, T_5$ are topological properties.